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Trigonometric identities, angular Schrödinger equations and a new family of solvable models

Vít Jakubský^{a,b,*}, Miloslav Znojil^a, Euclides Augusto Luís^c, Frieder Kleefeld^d

^a Nuclear Physics Institute, Řež, Czech Republic
 ^b FNSPE, CTU, Prague, Czech Republic
 ^c CEMAT, IST, Lisboa, Portugal
 ^d CFIF, IST, Lisboa, Portugal

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Abstract

Angular parts of certain solvable models are studied. We find that an extension of this class may be based on suitable trigonometric identities. The new exactly solvable Hamiltonians are shown to describe interesting two- and three-particle systems of the generalized Calogero, Wolfes and Winternitz–Smorodinsky types. © 2004 Elsevier B.V. All rights reserved.

1. Introduction

A family of superintegrable models has been introduced in [1]. One of them describes the movement of a particle in *D*-dimensional space and has the following Hamiltonian

$$H = \sum_{k=1}^{D} \left[-\frac{\partial^2}{\partial x_k^2} + \frac{\omega^2}{2} x_k^2 + \frac{g_k(g_k - 1)}{x_k^2} \right].$$
 (1)

The domain of definition is the set of functions which belong together with their first and second derivatives to $L^2(\bigotimes_{k=1}^{D} [(-\infty, 0) \cup (0, \infty)])$ and which vanish at $x_i = 0, i = 1, ..., D$. Due to its simplicity, Hamiltonian (1) serves as a useful playground for various methods in quantum mechanics. For D = 2, the pertaining Schrödinger

* Corresponding author.

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E-mail addresses: jakub@ujf.cas.cz (V. Jakubský), znojil@ujf.cas.cz (M. Znojil), eluis@math.ist.utl.pt (E.A. Luís), kleefeld@gtae3.ist.utl.pt (F. Kleefeld).

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equation is separable not only in Cartesian but also in polar and elliptical coordinates. In polar coordinates ($x_1 = r \cos \phi$, $x_2 = r \sin \phi$), the angular equation is

$$\Omega\xi_{l}(\phi) = \left[-\frac{\partial^{2}}{\partial\phi^{2}} + \frac{g_{1}(g_{1}-1)}{\sin^{2}\phi} + \frac{g_{2}(g_{2}-1)}{\cos^{2}\phi}\right]\xi_{l}(\phi) = b_{l}^{2}\xi(\phi),$$
(2)

where the solutions coincide with the well-known Pöschl-Teller [2] states defined in terms of Jacobi polynomials,

$$\xi_n^{(g_1,g_2)}(\phi) \sim \sin^{g_1}\phi \cos^{g_2}\phi P_n^{(g_1-\frac{1}{2},g_2-\frac{1}{2})}(\cos 2\phi),$$

$$b_l^2 = (g_1 + g_2 + 2n)^2.$$
(3)

In the most elementary special case with $g_1 = g$ and $g_2 = 0$ the domain of definition of our solutions is a union of two subdomains in the x_1-x_2 plane, separated by an impenetrable barrier. In the language of ϕ , we have to consider a union of two intervals, $Dom(\phi) = \bigcup_{k=0}^{1} (k\pi, (k+1)\pi)$.

Of course, a much more interesting model will work with both the couplings g_1 and g_2 different from zero where, at equal strengths $g_1 = g_2 = g \neq 0$, the wave functions are Gegenbauer polynomials,

$$\xi_n^{(g,g)}(\phi) \sim \sin^g 2\phi C_n^g(\cos 2\phi) \tag{4}$$

and coincide with eigenfunctions of the modification $-\frac{\partial^2}{\partial \phi^2} + \frac{4g(g-1)}{\sin^2 2\phi}$ of the operator Ω in (2). This coincidence is the consequence of the following simple trigonometric identity

$$\frac{1}{\sin^2 \phi} + \frac{1}{\cos^2 \phi} = \frac{4}{\sin^2 2\phi}.$$
(5)

The simplicity of the latter identity looks indicative and enigmatic at the same time. Firstly, it seems to reflect the separability of our problem as well as a symmetry in positions of the singularities on the circular domain, i.e., after some elementary trigonometry,

$$\frac{1}{\sin^2\phi} + \frac{1}{\sin^2(\phi - \pi/2)} + \frac{1}{\sin^2(\phi - \pi)} + \frac{1}{\sin^2(\phi - 3\pi/2)} = \frac{8}{\sin^2 2\phi}.$$
(6)

Secondly, the exceptionality of our choice of the identical strengths g might open an immediate relationship between the model (1) and several other solvable models based on the use of a suitable Lie algebra of symmetries [3] (cf. also Section 3.2 below and/or a very recent developments as sampled in Refs. [4–6]). Finally, our interest in the elementary trigonometry proved further enhanced by the recent independent clarification of the solvability of certain models using q-deformed Coxeter groups [7].

An overlap of all these observations formed a motivation of our forthcoming considerations.

2. Auxiliary trigonometric identities

It might be possible to find an identity resembling (5) when \sin^{-2} -type singularities cut the circular domain into N equal parts with an arbitrary integer N, $Dom(\phi) = \bigcup_{k=0}^{N-1} (\frac{2k\pi}{N}, \frac{2(k+1)\pi}{N})$. Such a desired generalization has been found to possess the form

$$\sum_{k=0}^{N-1} \frac{1}{\sin^2(\phi - \frac{2k\pi}{N})} = \begin{cases} \frac{N^2}{\sin^2 N\phi}, & N \text{ odd,} \\ \frac{N^2}{2\sin^2 \frac{N}{2}\phi}, & N \text{ even.} \end{cases}$$
(7)

The rigorous proof of its validity is both simple and straightforward. We start from

$$\sum_{k=0}^{N-1} \frac{1}{\sin^2(\phi - \frac{2\pi k}{N})} = -\frac{d^2}{d\phi^2} \ln\left(\prod_{k=0}^{N-1} \sin\left(\phi - \frac{2\pi k}{N}\right)\right)$$
(8)

and employ the known formula for the product of trigonometric functions [8],

$$\prod_{k=0}^{N-1} \sin\left(\phi - \frac{2\pi k}{N}\right) = \begin{cases} \frac{(-1)^{\frac{1-N}{2}}}{2^{N-1}} \sin N\phi, & N \text{ odd,} \\ \frac{(-1)^{\frac{-N}{2}}}{2^{N-1}} (1 - \cos N\phi), & N \text{ even.} \end{cases}$$
(9)

This gives Eq. (7) immediately. A cosine analog to (7)

$$\sum_{k=0}^{N-1} \frac{1}{\cos^2(\phi - \frac{2k\pi}{N})} = \begin{cases} \frac{N^2}{\cos^2 N\phi}, & N = 2p+1, \\ \frac{N^2}{2\cos^2 \frac{N}{2}\phi}, & N = 4p+2, \\ \frac{N^2}{2\sin^2 \frac{N}{2}\phi}, & N = 4p, \ p \in \mathbb{N}. \end{cases}$$
(10)

can be proved in the similar manner. We have to keep in mind that for the real arguments ϕ , the identities contain the periodic trigonometric functions. Even if we move to the complex arguments ϕ [9–11], the emergence of the phase of ϕ merely introduces a new rather artificial "degree of freedom" while the trigonometric identities themselves remain unchanged.

For the real ϕ we are now prepared to regard (2) as an N = 1 special case of a much broader class of the exactly solvable angular Schrödinger equations $\Omega_{(N)}\xi_m(\phi) = b_m^2\xi_m(\phi)$, i.e.,

$$\left(-\frac{\partial^2}{\partial\phi^2} + \sum_{k=0}^{N-1} \frac{g_1(g_1-1)}{\sin^2(\phi - \frac{2k\pi}{N})} + \sum_{l=0}^{N-1} \frac{g_2(g_2-1)}{\cos^2(\phi - \frac{2l\pi}{N})}\right) \xi_m(\phi) = b_m^2 \xi_m(\phi).$$
(11)

The eigenvalues are given by the following formulas for corresponding integers N

$$b_{m} = \frac{N}{4} \left(1 + \sqrt{1 + 8[g_{1}(g_{1} - 1) + g_{2}(g_{2} - 1)]} + 4m \right), \quad N = 4p,$$

$$b_{m} = \frac{N}{4} \left(2 + \sqrt{1 + 8[g_{1}(g_{1} - 1)]} + \sqrt{1 + 8[g_{2}(g_{2} - 1)]} + 4m \right), \quad N = 4p + 2,$$

$$b_{m} = N(g_{1} + g_{2} + 2m), \quad N = 2p + 1, \ p, m \in \mathbb{N}.$$
(12)

Also the solvability of (11) in terms of Jacobi polynomials is retained since we can perform the summations in the angular part V of the interaction,

$$V = \frac{N^2}{2} \left[\frac{g_1(g_1 - 1) + g_2(g_2 - 1)}{\sin^2 \frac{N}{2} \phi} \right], \quad N = 4p$$

$$= \frac{N^2}{2} \left[\frac{g_1(g_1 - 1)}{\sin^2 \frac{N}{2} \phi} + \frac{g_2(g_2 - 1)}{\cos^2 \frac{N}{2} \phi} \right], \quad N = 4p + 2$$

$$= N^2 \left[\frac{g_1(g_1 - 1)}{\sin^2 N \phi} + \frac{g_2(g_2 - 1)}{\cos^2 N \phi} \right], \quad N = 2p + 1, \ p \in \mathbb{N}$$
(13)

using the identities (7) and (10).

3. A few immediate applications

We may return back from Eq. (13) to the Pöschl–Teller bound-state problem in one dimension after an elementary linear transformation $\tilde{\phi} = N\phi$ or $\tilde{\phi} = \frac{N}{2}\phi$. Similarly, some of the related known separable and solvable models in more dimensions may be revealed as special cases as well. Nevertheless, our present key message is that in the latter context, also some new solvable models emerge due to our full freedom in the choice of the integer N in (11).

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3.1. One- and two-particle context

Once we leave the radial part of the (separable) partial differential equation (1) unchanged, our task is to perform just a backward transition to the original, "physical" Cartesian coordinates $y_1 = r \cos \phi$, $y_2 = r \sin \phi$. Introducing the fixed constant parameters $s_k = \sin \frac{2k\pi}{N}$ and $c_k = \cos \frac{2k\pi}{N}$ we get the new form of the Hamiltonian,

$$H = -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} + \frac{\omega^2}{2} \left(y_1^2 + y_2^2 \right) + \sum_{k=0}^{N-1} \frac{g_2(g_2 - 1)}{(y_1 c_k + y_2 s_k)^2} + \sum_{l=0}^{N-1} \frac{g_1(g_1 - 1)}{(y_1 s_l - y_2 c_l)^2},\tag{14}$$

where the mathematical separability of our general physical bound-state model remains "hidden". Its energies are easily expressible in the form $E_{n,m} = \sqrt{2}\omega(2n + \frac{\sqrt{2}}{2}b_m + 1)$ where the eigenvalues b_m of the angular equation itself are written explicitly in (12).

The latter formula may be read as characterizing a new and rather general two particle model where the interaction acquires the different forms for the different values of the integer N. In this way the most elementary choice of N = 1 returns us back to the Smorodinsky–Winternitz model (1).

At N = 2 the interaction looks much more complicated. Fortunately, after its brief inspection we reveal that it coincides with the so called BC_2 model of the exactly solvable type [3],

$$H_{BC_2} = \sum_{i=1}^{2} \left[-\frac{\partial^2}{\partial y_i^2} + \frac{\omega^2}{2} y_i^2 + \frac{g_1(g_1 - 1)}{y_i^2} \right] + \left(\frac{g_2(g_2 - 1)}{(y_2 - y_1)^2} + \frac{g_2(g_2 - 1)}{(y_2 + y_1)^2} \right).$$
(15)

In this sense we can consider (14) as a common generalization of the two mathematically different and apparently physically uncorrelated models.

3.2. Three-particle setting

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In the spirit of what has been said in Introduction, Hamiltonians (14) can easily be re-interpreted as describing *three* interacting particles on the line. It suffices to consider y_i as the two, so-called Jacobi coordinates of the system, to be complemented by the third, so-called center-of-mass (CMS) coordinate of the whole triplet. Formally, the transformation of the physical single-particle Cartesian coordinates x_1 , x_2 , x_3 into the CMS ones is given by the well-known formula,

$$\begin{pmatrix} y_1 \\ y_2 \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$
(16)

where Y is the coordinate of the center of mass. We add to (14) a kinetic and potential term $-\frac{\partial^2}{\partial Y^2} + \frac{\omega^2}{2}Y^2$ responsible, as usual, for the confined oscillatory motion of the center of mass of the whole system.

Using a transformation which is inverse to (16) we are now getting a new and interesting model of particles which interact via both an attractive and repulsive one-, two- and three-particle interaction in general. The new Hamiltonian reads

$$H = -\frac{1}{2} \frac{\partial^2}{\partial x_1^2} - \frac{1}{2} \frac{\partial^2}{\partial x_2^2} - \frac{1}{2} \frac{\partial^2}{\partial x_3^2} + \frac{1}{2} \omega^2 \left(x_1^2 + x_2^2 + x_3^2 \right) + \sum_{k=0}^{N-1} \frac{2g_2(g_2 - 1)}{[(c_k + \frac{\sqrt{3}}{3}s_k)x_1 + (-c_k + \frac{\sqrt{3}}{3}s_k)x_2 - \frac{2\sqrt{3}}{3}s_kx_3]^2} + \sum_{l=0}^{N-1} \frac{2g_1(g_1 - 1)}{[(-s_l + \frac{\sqrt{3}}{3}c_l)x_1 + (s_l + \frac{\sqrt{3}}{3}c_l)x_2 - \frac{2\sqrt{3}}{3}c_lx_3]^2}$$
(17)

and its spectrum is purely discrete. The explicit form of the energies

$$E_{n,m,t} = \sqrt{2}\omega\left(2n + \frac{\sqrt{2}}{2}b_m + 1\right) + \sqrt{2}\omega\left(t + \frac{1}{2}\right), \quad n, m, t \in \mathbb{N}$$

$$\tag{18}$$

contains a term which emerges due to an overall confinement of the whole triplet in the harmonic-oscillator well (= the contained movement of the center of the mass) and the term which depends directly on the angular-equation eigenvalue b_l (cf. (12)).

4. Discussion

We may summarize that in the context of Section 3.1 our new solvable Hamiltonians describe a particle which moves over a complicated potential surface containing also some strongly singular barriers. In the parallel angularequation re-interpretation of our trigonometric identities (and their consequences) in Section 3.2 we arrived at a new family of the genuine three-particle models which remain exactly solvable but which remain solely separable in the "unphysical", auxiliary polar coordinates.

A few more comments may be added.

4.1. The variability of N

Similarly to the generalized two-particle system (14), the present Hamiltonian (17) acquires the form of a well-known system at N = 3 and at the very special coupling strengths.

For $g_2 = 0$, Eq. (17) describes the well-known three-particle Calogero model [12], whose potential is

$$V_{\text{Cal}} = \frac{\omega^2}{2} \left(x_1^2 + x_2^2 + x_3^2 \right) + \sum_{j < k}^3 \frac{g_1(g_1 - 1)}{(x_k - x_j)^2}.$$

If we let $0 \neq g_1 \neq g_2 \neq 0$, the Wolfes model [13] with the following potential is revealed,

$$V_{\rm W} = \frac{\omega^2}{2} \left(x_1^2 + x_2^2 + x_3^2 \right) + \sum_{j < k}^3 \frac{g_1(g_1 - 1)}{(x_k - x_j)^2} + \sum_{l < m, l, m \neq n}^3 \frac{g_2(g_2 - 1)}{(x_l + x_m - 2x_n)^2}.$$
 (19)

In the next step let us evaluate the repulsive potential of (17) for a few higher values of the index N.

At N = 8, formula (13) suggests that the couplings will merge and form only one type of singular potential with coupling strength $g = (g_1(g_1 - 1) + g_2(g_2 - 1))$

$$V_{N=8} = g\left(\frac{4}{(x_1 - x_2)^2} + \frac{12}{(x_1 + x_2 - 2x_3)^2} + \sum_{\varepsilon = +, -} \frac{2}{[(\varepsilon \frac{1}{2} + \frac{\sqrt{3}}{3})x_1 + (-\varepsilon \frac{1}{2} + \frac{\sqrt{3}}{3})x_2 - \frac{\sqrt{3}}{3}x_3]^2}\right).$$
 (20)

A very different situation occurs at N = 5 where there emerge singularities of two types distinguished by the coupling strengths g_1 and g_2 ,

$$V_{N=5} = \sum_{k=1,2} \sum_{\varepsilon=+,-} \left[\frac{2g_2(g_2-1)}{[(c_k + \varepsilon \frac{\sqrt{3}}{3}s_k)x_1 + (-c_k + \varepsilon \frac{\sqrt{3}}{3}s_k)x_2 - \varepsilon \frac{2\sqrt{3}}{3}s_k x_3]^2} + \frac{2g_1(g_1-1)}{[(\frac{\sqrt{3}}{3}c_k - \varepsilon s_k)x_1 + (\frac{\sqrt{3}}{3}c_k + \varepsilon s_k)x_2 - \frac{2\sqrt{3}}{3}c_k x_3]^2} \right] + \frac{2g_2(g_2-1)}{(x_2 - x_1)^2} + \frac{6g_1(g_1-1)}{(x_1 + x_2 - 2x_3)^2}, \quad (21)$$

where one only has to keep in mind that $c_k = \cos \frac{k\pi}{5}$ and $s_k = \sin \frac{k\pi}{5}$.

4.2. Outlook

Being inspired by the simple superintegrable model (1), we derived the class of the trigonometric identities (7) and (10) by the entirely elementary mathematical means. These identities proved to be an unexpectedly productive tool for a generalization of several known solvable models. At the same time, the emergence and solvability of the new Hamiltonians opens many new questions.

We did not manage to touch many of them in this text. First of all, one must ask whether there exists an alternative or deeper algebraic background of their solvability. Next, even on a purely analytic level of our considerations a deeper insight would be welcome concerning the role of the repulsive barriers. Last but not least, a guidance towards a future analysis of the models with multiple barriers might be also sought somewhere in between their single-particle and multi-particle special cases. Thus, one would really appreciate seeing *more* formal parallels between the superintegrable, *separable* Winternitz–Smorodinsky-type systems in more dimensions and the intrinsically nonseparable *more-particle* systems belonging to the algebraically classified Calogero-type family.

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