hep-ph/9806415 19 Jun 1998

Trace evaluation of matrix determinants and inversion of 4×4 matrices in terms of Dirac covariants

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Abstract. In the following short paper we list some useful results concerning determinants and inverses of matrices. First we show, how to calculate determinants of $d \times d$ matrices, if their traces are known. As a next step 4×4 matrices are expressed in terms of Dirac covariants. The third step is the calculation of the corresponding inverse matrices in terms of Dirac covariants.

1 Calculation of matrix determinants and permanents in terms of their traces

We denote the determinant of a quadratic $d \times d$ matrix A by:

$$\det A = \left\| \begin{array}{ccc} A_{11} & \cdots & A_{1d} \\ \vdots & & \vdots \\ A_{d1} & \cdots & A_{dd} \end{array} \right\| \tag{1}$$

The calculation of the determinant of A for different dimensions d is straight forward:

$$\begin{aligned} d=1 \quad \Rightarrow \quad \det A &= A_{11} \\ d=2 \quad \Rightarrow \quad \det A &= A_{11}A_{22} - A_{12}A_{21} \\ d=3 \quad \Rightarrow \quad \det A &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - \\ &\quad - A_{31}A_{22}A_{13} - A_{32}A_{23}A_{11} - A_{33}A_{21}A_{12} \end{aligned}$$

[1] Preprint-No. FAU-TP3-97/4

$$\ldots \tag{2}$$

In a more unfamiliar way the determinants can be given by:

$$d=1 \;\Rightarrow\; \det A = \frac{1}{1!}\sum_{i=1}^{1}\left|\left|\; A_{ii} \;\right|\right| = \frac{1}{1!}\sum_{i=1}^{1} A_{ii}$$

$$d=2 \;\Rightarrow\; \det A = \frac{1}{2!}\sum_{i,j=1}^{2}\left|\left| \begin{array}{cc} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{array} \right|\right| = \frac{1}{2!}\sum_{i,j=1}^{2}\left(A_{ii}A_{jj}-A_{ij}A_{ji}\right)$$

$$d=3 \;\Rightarrow\; \det A = \frac{1}{3!}\sum_{i,j,k=1}^{3}\left|\left| \begin{array}{ccc} A_{ii} & A_{ij} & A_{ik} \\ A_{ji} & A_{jj} & A_{jk} \\ A_{ki} & A_{kj} & A_{kk} \end{array} \right|\right| =$$

$$= \frac{1}{3!}\sum_{i,j,k=1}^{3}\left(A_{ii}A_{jj}A_{kk}+A_{ij}A_{jk}A_{ki}+A_{ik}A_{ji}A_{kj}- \right.$$
$$\left. -A_{ki}A_{jj}A_{ik}-A_{kj}A_{jk}A_{ii}-A_{kk}A_{ji}A_{ij}\right)$$

$$d=4 \;\Rightarrow\; \det A = \frac{1}{4!}\sum_{i,j,k,\ell=1}^{4}\left|\left| \begin{array}{cccc} A_{ii} & A_{ij} & A_{ik} & A_{i\ell} \\ A_{ji} & A_{jj} & A_{jk} & A_{j\ell} \\ A_{ki} & A_{kj} & A_{kk} & A_{k\ell} \\ A_{\ell i} & A_{\ell j} & A_{\ell k} & A_{\ell\ell} \end{array} \right|\right| =$$

$$= \frac{1}{4!}\sum_{i,j,k,\ell=1}^{4}\left(A_{ii}A_{jj}A_{kk}A_{\ell\ell}\pm\ldots\right)$$

$$d=5 \;\Rightarrow\; \det A = \frac{1}{5!}\sum_{i,j,k,\ell,m=1}^{5}\left|\left| \begin{array}{ccccc} A_{ii} & A_{ij} & A_{ik} & A_{i\ell} & A_{im} \\ A_{ji} & A_{jj} & A_{jk} & A_{j\ell} & A_{jm} \\ A_{ki} & A_{kj} & A_{kk} & A_{k\ell} & A_{km} \\ A_{\ell i} & A_{\ell j} & A_{\ell k} & A_{\ell\ell} & A_{\ell m} \\ A_{mi} & A_{mj} & A_{mk} & A_{m\ell} & A_{mm} \end{array} \right|\right| =$$

$$= \frac{1}{5!}\sum_{i,j,k,\ell,m=1}^{5}\left(A_{ii}A_{jj}A_{kk}A_{\ell\ell}A_{mm}\pm\ldots\right)$$

$$\ldots \tag{3}$$

From these expansions it is easy to read off the following expressions which evaluate $\det A$ in terms of the $\mathrm{tr}A$:

$$d=1 \;\Rightarrow\; \det A = \frac{1}{1!}\,\mathrm{tr}A$$

$$d=2 \;\Rightarrow\; \det A = \frac{1}{2!}\left\{(\mathrm{tr}A)^2-\mathrm{tr}A^2\right\}$$

$$d=3 \;\Rightarrow\; \det A = \frac{1}{3!}\left\{(\mathrm{tr}A)^3-\mathrm{tr}A\;\mathrm{tr}A^2+2\;\mathrm{tr}A^3\right\}$$

$$d=4 \;\Rightarrow\; \det A = \frac{1}{4!}\left\{(\mathrm{tr}A)^4 - 6\,(\mathrm{tr}A)^2\,\mathrm{tr}A^2 + 8\,\mathrm{tr}A\,\mathrm{tr}A^3 + 3\,(\mathrm{tr}A^2)^2 - 6\,\mathrm{tr}A^4\right\}$$

$$d=5 \;\Rightarrow\; \det A = \frac{1}{5!}\left\{(\mathrm{tr}A)^5 - 10\,(\mathrm{tr}A)^3\,\mathrm{tr}A^2 + 20\,(\mathrm{tr}A)^2\,\mathrm{tr}A^3 - 30\,\mathrm{tr}A\,\mathrm{tr}A^4 - \right.$$
$$\left. -20\,\mathrm{tr}A^2\,\mathrm{tr}A^3 + 15\,\mathrm{tr}A\,(\mathrm{tr}A^2)^2 + 24\,\mathrm{tr}A^5\right\}$$

$$\ldots \tag{4}$$

For permanents the minus signs in the upper sums just have to be replaced by plus signs. In general we obtain the following formula for arbitrary dimension d:

$$\det A = \frac{1}{d!}\sum_{1k_1+2k_2+\ldots+dk_d=d} (-1)^{d-k_1-\ldots-k_d}\,\frac{d!}{1^{k_1}\ldots d^{k_d}\,k_1!\ldots k_d!}\left(\mathrm{tr}A^1\right)^{k_1}\ldots\left(\mathrm{tr}A^d\right)^{k_d}$$
$$(k_1,\ldots,k_d \in \{0,1,2,\ldots\}) \tag{5}$$

For permanents we obviously get:

$$\mathrm{per}\,A = \frac{1}{d!}\sum_{1k_1+2k_2+\ldots+dk_d=d} \frac{d!}{1^{k_1}\ldots d^{k_d}\,k_1!\ldots k_d!}\left(\mathrm{tr}A^1\right)^{k_1}\ldots\left(\mathrm{tr}A^d\right)^{k_d}$$
$$(k_1,\ldots,k_d \in \{0,1,2,\ldots\}) \tag{6}$$

The coefficients $d!/(1^{k_1}\ldots d^{k_d}\,k_1!\ldots k_d!)$ are discussed e.g. in [Lud96, p. 44].

2 Expansion of 4×4 matrices in terms of Dirac covariants

As a complete linear independent set of sixteen 4×4 matrices we can use the following so called Dirac covariants ($g^{\mu\nu}$ is the metric tensor):

$$\begin{aligned} &1_4 \\ &\gamma_5 \\ &\gamma^\mu \qquad (\mu=0,1,2,3), \\ &i\gamma_5\,\gamma^\mu \qquad (\mu=0,1,2,3) \\ &\sigma^{\mu\nu} \qquad (\mu<\nu)(\mu,\nu=0,1,2,3) \end{aligned} \tag{7}$$

($\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu := 2g^{\mu\nu}1_4$, $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma^{\mu\nu} = -\sigma^{\nu\mu} := \frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$ [Itz88, p. 692])

Apart from the 4 dimensional unit–matrix 1_4 all Dirac covariants are traceless. All Dirac covariants are trace orthogonal, i.e. if Γ and Γ' are two Dirac covariants we obtain:

$$\mathrm{tr}(\Gamma\Gamma') = 0 \quad \text{for} \quad \Gamma \neq \Gamma' \tag{8}$$

Very useful are the following traces:

$$\begin{aligned}
\mathrm{tr}(1_4 1_4) &= 4 \\
\mathrm{tr}(\gamma_5\gamma_5) &= 4 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4\,g^{\mu\nu} \\
\mathrm{tr}(i\gamma_5\gamma^\mu\, i\gamma_5\gamma^\nu) &= 4\,g^{\mu\nu} \\
\mathrm{tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) &= 4\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})
\end{aligned} \tag{9}$$

Using the notation $\not{a} := a_\mu\gamma^\mu$ (Einstein sum convention!) and the completeness of the Dirac covariants we can expand every 4×4 matrix M in terms of the Dirac covariants:

$$M = A\,1_4 + B\gamma_5 + \not{C} + i\gamma_5\not{D} + \frac{1}{2}E_{\mu\nu}\,\sigma^{\mu\nu} \quad (E_{\mu\nu} = -E_{\nu\mu}) \tag{10}$$

The coefficients are easily calculated by the trace properties of the Dirac covariants:

$$\begin{aligned}
A &= \frac{1}{4}\,\mathrm{tr}(M) \\
B &= \frac{1}{4}\,\mathrm{tr}(M\gamma_5) \\
C^\mu &= \frac{1}{4}\,\mathrm{tr}(M\gamma^\mu) \\
D^\mu &= \frac{1}{4}\,\mathrm{tr}(Mi\gamma_5\gamma^\mu) \\
E^{\mu\nu} &= \frac{1}{4}\,\mathrm{tr}(M\sigma^{\mu\nu})
\end{aligned} \tag{11}$$

Products, commutators and anticommutators of Dirac covariants evaluated and expanded in terms of Dirac covariants are given in appendix A.

3 Inversion of 4×4 matrices in terms of Dirac covariants

3.1 General case

Like in the previous section we consider an arbitrary 4×4 matrix expanded in terms of Dirac covariants:

$$M = A\,1_4 + B\gamma_5 + \not{C} + i\gamma_5\not{D} + \frac{1}{2}E_{\mu\nu}\,\sigma^{\mu\nu} \quad (E_{\mu\nu} = -E_{\nu\mu}) \tag{12}$$

Now we define the following shorthand notations:

$$E^*_{\mu\nu} := \varepsilon_{\mu\nu\rho\sigma}\,E^{\rho\sigma} \quad (\text{dual tensor of } E^{\mu\nu})$$

$$
\begin{aligned}
C \cdot D &:= C_\mu D^\mu \\
C^2 &:= C_\mu C^\mu \quad , \quad D^2 := D_\mu D^\mu \\
(D \pm iC)^2 &:= (D_\mu \pm iC_\mu)(D^\mu \pm iC^\mu) \\
E^2 &:= E_{\mu\nu} E^{\mu\nu} \\
E^* E &:= E^*_{\mu\nu} E^{\mu\nu} \quad =: E E^* \qquad (13)
\end{aligned}
$$

and observe the following identities:

$$(D-iC)^2(D+iC)^2 = (D^2-C^2)^2 + 4\,(C\cdot D)^2 = (D^2+C^2)^2 + 4\,(C\cdot D)^2 - 4\,C^2 D^2 \qquad (14)$$

$$\left(\frac{1}{2}\,\sigma_{\mu\nu} E^{\mu\nu}\right)^2 = \frac{1}{2}\,E^2 + \frac{i}{4}\,\gamma_5 E E^* \qquad (15)$$

$$[\not{C} - i\gamma_5 \not{D}]\,[\not{C} - i\gamma_5 \not{D}] = C^2 + D^2 + 2\gamma_5 \sigma_{\mu\nu}\, C^\mu D^\nu \qquad (16)$$

$$[\not{C} - i\gamma_5 \not{D}]\,[\not{C} + i\gamma_5 \not{D}] = C^2 - D^2 - 2i\,\gamma_5\, C\cdot D \qquad (17)$$

Using the trace formula (4) for $d = 4$ you can verify that:

$$
\begin{aligned}
\det M \quad = \\
&= (D-iC)^2(D+iC)^2 + \left[(A-B)^2 - \frac{1}{2}\,E^2 + \frac{i}{4} E E^*\right]\left[(A+B)^2 - \frac{1}{2}\,E^2 - \frac{i}{4} E E^*\right] - \\
&- 2\left\{(A^2 - B^2 + \frac{1}{2}\,E^2)(D^2+C^2) + 4\left[\frac{i}{2} A E^*_{\mu\nu} - B E_{\mu\nu}\right] C^\mu D^\nu - \right. \\
&- \left. 2\,E^\lambda{}_\mu E_{\lambda\nu}(C^\mu C^\nu + D^\mu D^\nu) - E^{*\,\lambda}{}_\mu E_{\lambda\nu}(C^\mu D^\nu - D^\mu C^\nu)\right\} \qquad (18)
\end{aligned}
$$

It is somewhat involved to derive the following expression for the inverse matrix M^{-1}:

$$
\begin{aligned}
M^{-1} = \frac{1}{\det M}\ \Bigg\{ &\left[(A + B\gamma_5 - \frac{1}{2}\,\sigma_{\mu\nu}\, E^{\mu\nu})(A - B\gamma_5) - \right. \\
&- \left. [\not{C} - i\gamma_5 \not{D}]\,[\not{C} - i\gamma_5 \not{D}]\right]\,[A - B\gamma_5 - \not{C} - i\gamma_5 \not{D}] - \\
&- (A + B\gamma_5 - \frac{1}{2}\,\sigma_{\mu\nu}\, E^{\mu\nu})\left[\frac{1}{2}\,E^2 - \frac{i}{4}\,\gamma_5 E E^* - [\not{C} + i\gamma_5 \not{D}]\,\frac{1}{2}\,\sigma_{\alpha\beta}\, E^{\alpha\beta}\right]
\end{aligned}
$$

$$- \quad [\not{C} - i\gamma_5 \not{D}] \frac{1}{2} \sigma_{\mu\nu} E^{\mu\nu} [\not{C} - i\gamma_5 \not{D}] \Bigg\} \tag{19}$$

For some purposes the following expression is more suitable:

$$\begin{aligned} M^{-1} = \frac{1}{\det M} \quad \Bigg\{ & \quad (A + B\gamma_5 - \frac{1}{2} \sigma_{\mu\nu} E^{\mu\nu}) \Bigg[(A - B\gamma_5)^2 - \frac{1}{2} E^2 + \frac{i}{4} \gamma_5 E E^* - \\ & - \quad [\not{C} + i\gamma_5 \not{D}] (A + B\gamma_5 - \frac{1}{2} \sigma_{\alpha\beta} E^{\alpha\beta}) \Bigg] - \\ & - \quad [\not{C} - i\gamma_5 \not{D}] \Bigg[(A + B\gamma_5 + \frac{1}{2} \sigma_{\alpha\beta} E^{\alpha\beta}) [\not{C} - i\gamma_5 \not{D}] - \\ & - \quad [\not{C} - i\gamma_5 \not{D}] [\not{C} + i\gamma_5 \not{D}] \Bigg] \Bigg\} \end{aligned} \tag{20}$$

3.2 The case $E^{\mu\nu} = 0$

In this subsection we take the limit $E^{\mu\nu} = 0$, i.e.:

$$M = A\, 1_4 + B\gamma_5 + \not{C} + i\gamma_5 \not{D} \tag{21}$$

In this limit we obtain for $\det M$:

$$\begin{aligned} \det M \quad & = \\ & = \quad (A^2 - B^2 - C^2 - D^2)^2 + 4\,(C \cdot D)^2 - 4\,C^2 D^2 = \\ & = \quad (A^2 - B^2)^2 - 2\,(A^2 - B^2)(C^2 + D^2) + (C^2 + D^2)^2 + 4\,(C \cdot D)^2 - 4\,C^2 D^2 = \\ & = \quad (A^2 - B^2)^2 - 2\,(A^2 - B^2)(C^2 + D^2) + (C^2 - D^2)^2 + 4\,(C \cdot D)^2 = \\ & = \quad (A^2 - B^2)^2 - 2\,(A^2 - B^2)(C^2 + D^2) + (C - iD)^2 (C + iD)^2 = \\ & = \quad (A^2 - B^2 - 2C^2 - 2C^2)(A^2 - B^2) + (C - iD)^2 (C + iD)^2 \end{aligned} \tag{22}$$

The corresponding inverse matrix M^{-1} is:

$$\begin{aligned} M^{-1} \quad & = \frac{1}{\det M} \quad \Big[A^2 - B^2 - [\not{C} - i\gamma_5 \not{D}] [\not{C} - i\gamma_5 \not{D}] \Big] [A - B\gamma_5 - \not{C} - i\gamma_5 \not{D}] \\ & = \frac{1}{\det M} \quad \Big[A^2 - B^2 - C^2 - D^2 - 2\gamma_5 \sigma_{\mu\nu} C^\mu D^\nu \Big] [A - B\gamma_5 - \not{C} - i\gamma_5 \not{D}] \end{aligned} \tag{23}$$

If C^μ is proportional to D^μ, i.e. $C^\mu \sim D^\mu$, we get significant simplifications. We note that:

$$C^\mu \sim D^\mu \quad \Rightarrow \quad 4\,(C\cdot D)^2 - 4\,C^2 D^2 = 0 \quad , \quad \sigma_{\mu\nu}\, C^\mu D^\nu = 0 \tag{24}$$

i.e.

$$C^\mu \sim D^\mu \quad \Rightarrow \quad \det M = (A^2 - B^2 - C^2 - D^2)^2 \tag{25}$$

For the inverse matrix we obtain:

$$\begin{aligned} C^\mu \sim D^\mu \quad \Rightarrow \quad M^{-1} &= \frac{1}{\det M}(A^2 - B^2 - C^2 - D^2)\,[A - B\gamma_5 - \not{C} - i\gamma_5\,\not{D}] = \\ &= \frac{A - B\gamma_5 - \not{C} - i\gamma_5\,\not{D}}{A^2 - B^2 - C^2 - D^2} \end{aligned} \tag{26}$$

Some special examples with respect to this case are:

$$\begin{aligned} (A + B\gamma_5 + \not{C})^{-1} &= \frac{A - B\gamma_5 - \not{C}}{A^2 - B^2 - C^2} \\ (A + B\gamma_5 + i\gamma_5\,\not{D})^{-1} &= \frac{A - B\gamma_5 - i\gamma_5\,\not{D}}{A^2 - B^2 - D^2} \end{aligned} \tag{27}$$

Another interesting situation is the case $A = \pm B$, i.e. we start from the matrix:

$$M = A\,(1_4 \pm \gamma_5) + \not{C} + i\gamma_5\,\not{D} \tag{28}$$

Then we get for $\det M$:

$$\det M \quad = \quad (C - iD)^2 (C + iD)^2 \tag{29}$$

The inverse M^{-1} is given by:

$$M^{-1} = \frac{-1}{\det M}\;\; [\,\not{C} - i\gamma_5\,\not{D}]\;[\,\not{C} - i\gamma_5\,\not{D}]\;[A\,(1_4 \mp \gamma_5) - \not{C} - i\gamma_5\,\not{D}] \tag{30}$$

3.3 The case $C^\mu = D^\mu = 0$

In this subsection we take the limit $C^\mu = D^\mu = 0$, i.e.:

$$M = A\,1_4 + B\gamma_5 + \frac{1}{2}\,E_{\mu\nu}\,\sigma^{\mu\nu} \quad (E_{\mu\nu} = -E_{\nu\mu}) \tag{31}$$

In this limit we obtain for $\det M$:

$$\det M \quad = \quad \left[(A - B)^2 - \frac{1}{2}\,E^2 + \frac{i}{4} E E^*\right]\,\left[(A + B)^2 - \frac{1}{2}\,E^2 - \frac{i}{4} E E^*\right] \tag{32}$$

The inverse matrix M^{-1} is given by:

$$M^{-1} = \frac{1}{\det M}\left(A+B\gamma_5-\frac{1}{2}\,\sigma_{\mu\nu}\,E^{\mu\nu}\right)\left[(A-B\gamma_5)^2-\frac{1}{2}\,E^2+\frac{i}{4}\,\gamma_5 EE^*\right]=$$
$$= \frac{1}{\det M}\left[(A-B\gamma_5)^2-\frac{1}{2}\,E^2+\frac{i}{4}\,\gamma_5 EE^*\right]\left(A+B\gamma_5-\frac{1}{2}\,\sigma_{\mu\nu}\,E^{\mu\nu}\right) \tag{33}$$

A Products, commutators and anticommutators of Dirac covariants

The totally antisymmetric Levi-Civita tensor is defined by [Itz88, p. 692]:

$$\varepsilon_{\mu\nu\rho\sigma} := \left\| \begin{array}{cccc} g_{\mu 0} & g_{\mu 1} & g_{\mu 2} & g_{\mu 3} \\ g_{\nu 0} & g_{\nu 1} & g_{\nu 2} & g_{\nu 3} \\ g_{\rho 0} & g_{\rho 1} & g_{\rho 2} & g_{\rho 3} \\ g_{\sigma 0} & g_{\sigma 1} & g_{\sigma 2} & g_{\sigma 3} \end{array} \right\| \tag{34}$$

For the commutator and anticommutator of A and B we write:

$$[A,B] := AB-BA \quad , \quad \{A,B\} := AB+BA \tag{35}$$

Some calculation yields the following expansions of products of Dirac covariants in terms of Dirac covariants ($\varepsilon^{\mu\nu}{}_{\lambda\lambda'}\,\varepsilon^{\lambda\lambda'\rho\sigma} \stackrel{!}{=} -2\,(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})$):

$1_4\cdot 1_4 = 1_4$	$1_4\cdot\gamma_5 = \gamma_5$	$1_4\cdot\gamma_\mu = \gamma_\mu$
$\gamma_5\cdot 1_4 = \gamma_5$	$\gamma_5\cdot\gamma_5 = 1_4$	$\gamma_5\cdot\gamma_\mu = -i(i\gamma_5\gamma_\mu)$
$\gamma_\mu\cdot 1_4 = \gamma_\mu$	$\gamma_\mu\cdot\gamma_5 = i(i\gamma_5\gamma_\mu)$	$\gamma_\mu\cdot\gamma_\nu = g_{\mu\nu}1_4 - i\sigma_{\mu\nu}$
$i\gamma_5\gamma_\mu\cdot 1_4 = i\gamma_5\gamma_\mu$	$i\gamma_5\gamma_\mu\cdot\gamma_5 = -i\gamma_\mu$	$i\gamma_5\gamma_\mu\cdot\gamma_\nu = i\,g_{\mu\nu}\gamma_5+\frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma}$
$\sigma_{\mu\nu}\cdot 1_4 = \sigma_{\mu\nu}$	$\sigma_{\mu\nu}\cdot\gamma_5 = \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma}$	$\sigma_{\mu\nu}\cdot\gamma_\rho = i\varepsilon_{\sigma\mu\nu\rho}\,i\gamma_5\gamma^\sigma + \frac{i}{2}\,\varepsilon_{\mu\nu\lambda\lambda'}\,\varepsilon^{\lambda\lambda'}{}_{\rho\sigma}\,\gamma^\sigma$

$1_4 \cdot i\gamma_5\gamma_\mu = i\gamma_5\gamma_\mu$	$1_4 \cdot \sigma_{\mu\nu} = \sigma_{\mu\nu}$
$\gamma_5 \cdot i\gamma_5\gamma_\mu = i\gamma_\mu$	$\gamma_5 \cdot \sigma_{\mu\nu} = \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma}$
$\gamma_\mu \cdot i\gamma_5\gamma_\nu = -ig_{\mu\nu}\gamma_5 - \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma}$	$\gamma_\rho \cdot \sigma_{\mu\nu} = i\,\varepsilon_{\sigma\mu\nu\rho}\,i\gamma_5\gamma^\sigma -$ $- \frac{i}{2}\,\varepsilon_{\mu\nu\lambda\lambda'}\,\varepsilon^{\lambda\lambda'}{}_{\rho\sigma}\,\gamma^\sigma$
$i\gamma_5\gamma_\mu \cdot i\gamma_5\gamma_\nu = g_{\mu\nu}1_4 - i\sigma_{\mu\nu}$	$i\gamma_5\gamma_\rho \cdot \sigma_{\mu\nu} = -i\varepsilon_{\sigma\mu\nu\rho}\,\gamma^\sigma -$ $- \frac{i}{2}\,\varepsilon_{\mu\nu\lambda\lambda'}\,\varepsilon^{\lambda\lambda'}{}_{\rho\sigma}\,i\gamma_5\gamma^\sigma$
$\sigma_{\mu\nu} \cdot i\gamma_5\gamma_\rho = -i\varepsilon_{\sigma\mu\nu\rho}\,\gamma^\sigma +$ $+ \frac{i}{2}\,\varepsilon_{\mu\nu\lambda\lambda'}\,\varepsilon^{\lambda\lambda'}{}_{\rho\sigma}\,i\gamma_5\gamma^\sigma$	$\sigma_{\mu\nu} \cdot \sigma_{\rho\sigma} = (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,1_4 +$ $+ i\,(\sigma_{\mu\sigma}g_{\nu\rho} - \sigma_{\mu\rho}g_{\nu\sigma} +$ $+ g_{\mu\sigma}\sigma_{\nu\rho} - g_{\mu\rho}\sigma_{\nu\sigma}) +$ $+ i\,\varepsilon_{\mu\nu\rho\sigma}\,\gamma_5$

(36)

The commutators of the Dirac covariants are evaluated to be:

$[1_4, 1_4] = 0$	$[1_4, \gamma_5] = 0$	$[1_4, \gamma_\mu] = 0$
$[\gamma_5, 1_4] = 0$	$[\gamma_5, \gamma_5] = 0$	$[\gamma_5, \gamma_\mu] = -2i(i\gamma_5\gamma_\mu)$
$[\gamma_\mu, 1_4] = 0$	$[\gamma_\mu, \gamma_5] = 2i(i\gamma_5\gamma_\mu)$	$[\gamma_\mu, \gamma_\nu] = -2i\sigma_{\mu\nu}$
$[i\gamma_5\gamma_\mu, 1_4] = 0$	$[i\gamma_5\gamma_\mu, \gamma_5] = -2i\gamma_\mu$	$[i\gamma_5\gamma_\mu, \gamma_\nu] = 2i\,g_{\mu\nu}\gamma_5$
$[\sigma_{\mu\nu}, 1_4] = 0$	$[\sigma_{\mu\nu}, \gamma_5] = 0$	$[\sigma_{\rho\sigma}, \gamma_\mu] = 2i\,(\gamma_\rho g_{\mu\sigma} - \gamma_\sigma g_{\mu\rho})$

$[1_4, i\gamma_5\gamma_\mu] = 0$	$[1_4, \sigma_{\mu\nu}] = 0$
$[\gamma_5, i\gamma_5\gamma_\mu] = 2i\gamma_\mu$	$[\gamma_5, \sigma_{\mu\nu}] = 0$
$[\gamma_\mu, i\gamma_5\gamma_\nu] = -2ig_{\mu\nu}\gamma_5$	$[\gamma_\mu, \sigma_{\rho\sigma}] = 2i\,(\gamma_\sigma g_{\mu\rho} - \gamma_\rho g_{\mu\sigma})$
$[i\gamma_5\gamma_\mu, i\gamma_5\gamma_\nu] = -2i\sigma_{\mu\nu}$	$[i\gamma_5\gamma_\mu, \sigma_{\rho\sigma}] = 2i\,(i\gamma_5\gamma_\sigma g_{\mu\rho} - i\gamma_5\gamma_\rho g_{\mu\sigma})$
$[\sigma_{\rho\sigma}, i\gamma_5\gamma_\mu] = 2i\,(i\gamma_5\gamma_\rho g_{\mu\sigma} - i\gamma_5\gamma_\sigma g_{\mu\rho})$	$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = 2i\,(\sigma_{\mu\sigma}g_{\nu\rho} - \sigma_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}\sigma_{\nu\rho} - g_{\mu\rho}\sigma_{\nu\sigma})$

(37)

For the anticommutators of the Dirac covariants we obtain:

$\{1_4, 1_4\} = 2\,1_4$	$\{1_4, \gamma_5\} = 2\,\gamma_5$	$\{1_4, \gamma_\mu\} = 2\,\gamma_\mu$
$\{\gamma_5, 1_4\} = 2\,\gamma_5$	$\{\gamma_5, \gamma_5\} = 2\,1_4$	$\{\gamma_5, \gamma_\mu\} = 0$
$\{\gamma_\mu, 1_4\} = 2\,\gamma_\mu$	$\{\gamma_\mu, \gamma_5\} = 0$	$\{\gamma_\mu, \gamma_\nu\} = 2\,g_{\mu\nu}\,1_4$
$\{i\gamma_5\gamma_\mu, 1_4\} = 2i\gamma_5\gamma_\mu$	$\{i\gamma_5\gamma_\mu, \gamma_5\} = 0$	$\{i\gamma_5\gamma_\mu, \gamma_\nu\} = i\,\varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma}$
$\{\sigma_{\mu\nu}, 1_4\} = 2\,\sigma_{\mu\nu}$	$\{\sigma_{\mu\nu}, \gamma_5\} = i\,\varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma}$	$\{\sigma_{\mu\nu}, \gamma_\rho\} = 2i\varepsilon_{\sigma\mu\nu\rho}\,i\gamma_5\gamma^\sigma$

$\{1_4, i\gamma_5\gamma_\mu\} = 2i\,\gamma_5\gamma_\mu$	$\{1_4, \sigma_{\mu\nu}\} = 2\,\sigma_{\mu\nu}$
$\{\gamma_5, i\gamma_5\gamma_\nu\} = 0$	$\{\gamma_5, \sigma_{\mu\nu}\} = i\,\varepsilon_{\mu\nu\rho\sigma}\ \sigma^{\rho\sigma}$
$\{\gamma_\mu, i\gamma_5\gamma_\nu\} = -i\,\varepsilon_{\mu\nu\rho\sigma}\ \sigma^{\rho\sigma}$	$\{\gamma_\rho, \sigma_{\mu\nu}\} = 2i\,\varepsilon_{\sigma\mu\nu\rho}\ i\gamma_5\gamma^\sigma$
$\{i\gamma_5\gamma_\mu, i\gamma_5\gamma_\nu\} = 2g_{\mu\nu}\,1_4$	$\{i\gamma_5\gamma_\rho, \sigma_{\mu\nu}\} = -2i\,\varepsilon_{\sigma\mu\nu\rho}\ \gamma^\sigma$
$\{\sigma_{\mu\nu}, i\gamma_5\gamma_\rho\} = -2i\,\varepsilon_{\sigma\mu\nu\rho}\ \gamma^\sigma$	$\{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} = 2\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,1_4 + 2i\,\varepsilon_{\mu\nu\rho\sigma}\ \gamma_5$

(38)

References

[Itz88] C. Itzykson, J.-B. Zuber:
Quantum field theory
(4th printing 1988)
©McGraw–Hill, Inc., 1980

[Lud96] W. Ludwig, C. Falter:
Symmetries in physics (Group theory applied to physical problems)
(Second edition)
©Springer–Verlag Berlin Heidelberg, 1996