arXiv:hep-ph/0211275 v1 18 Nov 2002

Meson Form Factors and the Quark-Level Linear σ Model

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PACS numbers: 12.40.-y, 11.30.Rd, 13.40.Gp, 13.40.Hq, 13.20.Cz, 13.20.Eb, 13.25.Es, 12.40.Vv

hep-ph/0211275

November 18, 2002

Abstract

The quark-level linear σ model (LσM) is employed to compute a variety of electromagnetic and weak observables of light mesons, including pion and kaon form factors and charge radii, charged-pion polarizabilities, semileptonic weak $K_{\ell 3}$ decay, semileptonic weak radiative pion and kaon form factors, radiative decays of vector mesons, and nonleptonic weak $K_{2\pi}$ decay. The agreement of all these predicted observables with experiment is striking. In passing, the tight link between the LσM and vector-meson dominance is shown. Some conclusions are drawn on the LσM in connection with lattice and renormalization-group approaches to QCD.

1 Survey of LσM and chiral Goldberger–Treiman relations

For the past eight years, there has been much experimental [1] and theoretical [2] activity, as well as combined workshops [3], concerning isoscalar scalar mesons in general, and the σ meson in particular. Very recently, we have employed electromagnetic (e.m.) and weak processes to conclude that the mostly nonstrange $\bar{n}n$ resonances are the $f_0(600)$ and the $f_0(1370)$, while the $f_0(980)$ and the $f_0(1500)$ are mainly $\bar{s}s$ [4]. In the present paper, we study meson (π,K) form factors in general, and specialize at a later stage to a specific scheme, namely the quark-level linear σ model (LσM).

Nonperturbatively solving [5] the strong-interaction Nambu-type gap equations $\delta f_\pi = f_\pi$ and $\delta\hat{m} = m$ (where f_π is the pion decay constant and $\hat{m}$ is the nonstrange constituent quark mass) in

quark-loop order, regularization schemes lead to the NJL [6] and $Z = 0$ compositeness [7] relations

$$m_\sigma \;=\; 2\hat{m} \quad , \quad g \;=\; \frac{2\pi}{\sqrt{N_c}} \; , \tag{1}$$

with $\hat{m} \sim M_N/3$ and meson-quark coupling $g = 2\pi/\sqrt{3} = 3.628$. For a more detailed description of the quark-level LσM, see the Appendix. Here, we survey instead meson form factors and related data in a LσM context for strong, e.m., and weak interactions.

This chiral LσM is based on the quark-level pion and kaon Goldberger–Treiman relations (GTRs)

$$f_\pi \, g \;=\; \hat{m} \;=\; \frac{1}{2}(m_u + m_d) \quad , \quad f_K \, g \;=\; \frac{1}{2}(m_s + \hat{m}) \; , \tag{2}$$

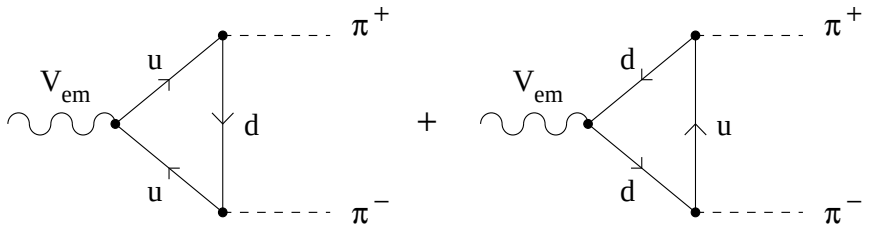
for $f_\pi \approx 93$ MeV ($f_\pi \approx 90$ MeV in the chiral limit (CL) [8]), $f_K/f_\pi \approx 1.22$, and $m_s \approx 1.44\,\hat{m}$ (from Eq. (2)). We begin in Sec. 2 by studying meson vector form factors and their measured charge radii. In Sec. 3 we survey charged-pion polarizabilities for $\gamma\gamma \to \pi\pi$, and compare the results with the LσM predictions. In Sec. 4 we study the semileptonic weak K_{l3} decays and the form factor $f_+(k^2)$ evaluated at $k^2 = 0$. Then in Sec. 5 we examine the radiative semileptonic weak form factors for $\pi^+ \to e^+\nu\gamma$ and $K^+ \to e^+\nu\gamma$ decays, with the observed pion second axial-vector form factor implying a pion charge radius $r_\pi \sim 0.6$ fm, also found in Sec. 2 from data [9] and from the theoretical LσM. In Sec. 6 we return to the LσM and its link with vector-meson dominance (VMD). Finally, in Sec. 7 we begin by studying the $\Delta I = 1/2$ rule for two-pion decays of the kaon in connection with the σ as the pion's chiral partner, and end by showing that the mass of the now experimentally confirmed scalar κ meson is consistent with the observed $K \to 2\pi$ decay rate. We summarize our results and draw our conclusions in Sec. 8.

2 Meson vector form factors and charge radii

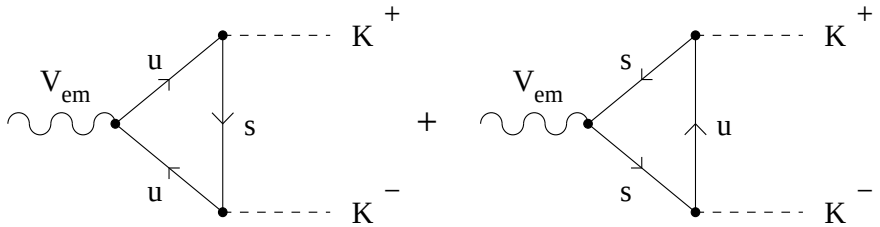
The charged-pion and kaon e.m. vector currents are defined as

$$\begin{aligned} \Big\langle \pi^+(q') | V^\mu_{em}(0) | \pi^+(q) \Big\rangle \;&=\; F_\pi(k^2) \; (q'+q)^\mu \; , \\ \Big\langle K^+(q') | V^\mu_{em}(0) | K^+(q) \Big\rangle \;&=\; F_K(k^2) \; (q'+q)^\mu \; , \end{aligned} \tag{3}$$

with $k_\mu = q'_\mu - q_\mu$. The former pion form factor $F_\pi(k^2)$ can be — perturbatively — characterized by the (constituent) quark udu and dud loop graphs of Fig. 1a, while the charged-kaon form factor $F_K(k^2)$ is in a similar manner determined by the usu and sus loop graphs depicted in Fig. 1b. Even if each of the diagrams in Fig. 1 appears to be linearly divergent by naive power counting, gauge invariance enforces every single quark triangle (QT) to be merely logarithmically divergent. After evaluation of spin traces, the form factors in Eq. (3) can be brought to the form (with color



(a)



(b)

Figure 1: VPP quark triangle graphs.

number $N_c = 3$)

$$F_\pi(k^2)_{\rm QT} = -4ig^2N_c\left(+\frac{2}{3}\,I(k^2,m_u^2,m_d^2,m_\pi^2)+\frac{1}{3}\,I(k^2,m_d^2,m_u^2,m_\pi^2)\right)\,, \tag{4}$$

$$F_K(k^2)_{\rm QT} = -4ig^2N_c\left(+\frac{2}{3}\,I(k^2,m_u^2,m_s^2,m_K^2)+\frac{1}{3}\,I(k^2,m_s^2,m_u^2,m_K^2)\right)\,. \tag{5}$$

The integral $I(k^2,m_q^2,m_{_Q}^2,M^2)$ is defined by

$$\begin{aligned} I(k^2,m_q^2,m_{_Q}^2,M^2) &= \\ &= \frac{i\pi^2}{(2\pi)^4}\,\frac{1}{2}\int_0^1 dv\int_v^1 du\,\frac{k^2u+2\,(M^2-(m_q-m_{_Q})^2)(1-u)}{m_Q^2-\left(M^2+m_Q^2-m_q^2\right)u+M^2u^2+\frac{1}{4}\,(v^2-u^2)k^2} \\ &\quad+\int_0^1 dx\int \bar{d}^4p\left[p^2-m_Q^2+\left(M^2+m_Q^2-m_q^2\right)x-M^2x^2\right]^{-2}\,, \end{aligned} \tag{6}$$

where $\bar{d}^4p = d^4p\,(2\pi)^{-4}$.

The perturbative QT expressions (4)–(6) in the CL (i.e. $M\to 0$) should be compared to a CL non-perturbative LσM result [5, 10]

$$F_\pi(k^2)^{\rm CL}_{\rm L\sigma M} = -4ig^2N_c\int_0^1 dx\int \bar{d}^4p\left[p^2-\hat{m}^2+x(1-x)k^2\right]^{-2}\,, \tag{7}$$

$$F_K(k^2)^{\rm CL}_{\rm L\sigma M} = -4ig^2N_c\int_0^1 dx\int \bar{d}^4p\left[p^2-m_{us}^2+x(1-x)k^2\right]^{-2}\,, \tag{8}$$

where $m_{us} = (m_s+\hat{m})/2$. The logarithmic divergence of these expressions has been guaranteed through a rerouting procedure [10, 11]. When $k^2 = 0$, these form factors become automatically

normalized to unity, i.e.,

$$F_\pi(0)^{\rm CL}_{\rm L\sigma M} \;=\; -4ig^2N_c\int \not{d}^4p\,\left[p^2-\hat{m}^2\right]^{-2} \;=\; 1\;, \tag{9}$$

$$F_K(0)^{\rm CL}_{\rm L\sigma M} \;=\; -4ig^2N_c\int \not{d}^4p\,\left[p^2-m_{us}^2\right]^{-2} \;=\; 1\;, \tag{10}$$

due to the GTRs in Eq. (2), and the definition of the pion and kaon decay constants $\langle 0|A_3^\mu|\pi^0\rangle = if_\pi q^\mu$, $\langle 0|A_{4-i5}^\mu|K^+\rangle = i\sqrt{2}f_Kq^\mu$, with $f_\pi\approx 93$ MeV and $f_K/f_\pi\approx 1.22$ [5, 11].

In contrast, the perturbative QT results yield in the CL

$$F_{\pi^+}(0)^{\rm CL}_{\rm QT} \;=\; -4ig^2N_c\int \not{d}^4p\,\left[p^2-\hat{m}^2\right]^{-2}\;, \tag{11}$$

$$\begin{aligned} F_{K^+}(0)^{\rm CL}_{\rm QT} \;=\; & -4ig^2N_c\left\{\int \not{d}^4p\,\left[\left(p^2-\hat{m}^2\right)\left(p^2-m_s^2\right)\right]^{-1}\right. \\ & \left. -\frac{i\pi^2}{(2\pi)^4}\,\frac{1}{2\,(m_s+\hat{m})^2}\left(m_s^2+\hat{m}^2-\frac{2m_s^2\hat{m}^2}{m_s^2-\hat{m}^2}\,\ln\left|\frac{m_s^2}{\hat{m}^2}\right|\right)\right\}\;, \end{aligned} \tag{12}$$

being — up to an finite constant correction term in the case of the kaon — normalized by the logarithmically divergent gap equations (LDGEs) (see Ref. [2], seventh paper)

$$1 \;=\; -4ig^2N_c\int \not{d}^4p\,\left[p^2-\hat{m}^2\right]^{-2}\;, \tag{13}$$

$$1 \;=\; -4ig^2N_c\int \not{d}^4p\,\left[\left(p^2-\hat{m}^2\right)\left(p^2-m_s^2\right)\right]^{-1}\;. \tag{14}$$

To proceed, given Eqs. (7) and (8), the meson charge radii are computed in the LσM as

$$\begin{aligned} \left\langle r^2_{\pi^+}\right\rangle^{\rm CL}_{\rm L\sigma M} \;=\; 6\,\frac{dF_\pi(k^2)}{dk^2}\bigg|_{k^2=0} \;&=\; \frac{-i4N_cg^2\,(-2)}{(2\pi)^4}\int_0^1 dx\,6x(1-x)\int \not{d}^4p\,\left[p^2-\hat{m}^2\right]^{-3} \\ &=\; \frac{8iN_c}{(2\pi)^4}\,g^2\,(\frac{-i\pi^2}{2\hat{m}^2}) \;=\; \frac{N_c}{4\pi^2f_\pi^2} \;\approx\; (0.61\,{\rm fm})^2 \end{aligned} \tag{15}$$

and (the obvious $SU(3)$ extension)

$$\begin{aligned} \left\langle r^2_{K^+}\right\rangle^{\rm CL}_{\rm L\sigma M} \;=\; 6\,\frac{dF_K(k^2)}{dk^2}\bigg|_{k^2=0} \;&=\; \frac{-i4N_cg^2\,(-2)}{(2\pi)^4}\int_0^1 dx\,6x(1-x)\int \not{d}^4p\,\left[p^2-m_{us}^2\right]^{-3} \\ &=\; \frac{8iN_c}{(2\pi)^4}\,g^2\,(\frac{-i\pi^2}{2m_{us}^2}) \;=\; \frac{N_c}{4\pi^2f_K^2} \;\approx\; (0.49\,{\rm fm})^2\,. \end{aligned} \tag{16}$$

Here we have evaluated the charge radii in the CL [5, 8, 12], with $f_\pi^{\rm CL}\approx 90$ MeV, $f_K^{\rm CL}\approx 110$ MeV.

At this point we may return to the perturbative QT results (4) and (5), from which we derive in the CL

$$\left\langle r^2_{\pi^+}\right\rangle^{\rm CL}_{\rm QT} \;=\; \frac{g^2N_c}{4\pi^2\hat{m}^2} \;\stackrel{!}{=}\; \left\langle r^2_{\pi^+}\right\rangle^{\rm CL}_{\rm L\sigma M}\;, \tag{17}$$

$$\left\langle r^2_{K^+} \right\rangle^{\rm CL}_{\rm QT} = \frac{g^2 N_c}{4\pi^2 m^2_{us}}$$

$$\times \frac{1}{12} \frac{1}{(m_s^2 - \hat{m}^2)^2} \left\{ \hat{m}^4 - 3\hat{m}^3 m_s - 5\hat{m}^2 m_s^2 - 15 \hat{m} m_s^3 - 2 m_s^4 \right.$$

$$\left. + 2 \frac{\hat{m}^6 + 3\hat{m}^5 m_s + 6 \hat{m} m_s^5 + 2 m_s^6}{m_s^2 - \hat{m}^2} \ln \left| \frac{m_s^2}{\hat{m}^2} \right| \right\}$$

$$= \frac{g^2 N_c}{4\pi^2 \hat{m}^2} \left(1 - \frac{5}{6}\delta + \frac{3}{5}\delta^2 - \frac{4}{9}\delta^3 + \frac{22}{63}\delta^4 - \frac{2}{7}\delta^5 + \ldots \right) , \tag{18}$$

with $m_s = (1+\delta)\hat{m}$, i.e., $\delta = (m_s/\hat{m}) - 1 \approx 0.44$. The coefficients of the presented Taylor expansion in the SU(3)-breaking parameter δ coincide with the ones given in Ref. [13], while our full result is also in agreement with the expressions originally derived by Tarrach [14]. Taking into account the first three terms of this expansion, we may estimate the ratio r_K/r_π to be

$$\frac{\langle r^2_{K^+} \rangle}{\langle r^2_{\pi^+} \rangle} \approx 1 - \frac{5}{6}\,\delta + \frac{3}{5}\,\delta^2 \approx 0.750 \quad \text{or} \quad \frac{\langle r_{K^+} \rangle}{\langle r_{\pi^+} \rangle} \approx 0.866 \, . \tag{19}$$

Here we note that the observed pion charge radius is [9]

$$r_\pi = (0.642 \pm 0.002) \text{ fm} , \tag{20}$$

and the analogue charged-kaon charge radius is [1]

$$r_K = (0.560 \pm 0.031) \text{ fm} . \tag{21}$$

If we take the experimental value $r_{\pi^+} \approx 0.64$ fm from Eq. (20), the latter ratio (19) implies $< r_{K^+} > \approx 0.556$ fm, which is compatible with Eq. (21).

In summary, the more detailed perturbative results of Eqs. (4), (5), (6), (11), (12), (17), and (18) are compatible with the simpler non-perturbative (SU(3)-symmetry) scheme of Eqs. (7)–(10), (15), and (16) above. Thus, no further renormalization needs be considered in either case. Note, too, that these detailed or simple field-theory versions of the charged-pion form factor can be recovered in an even simpler fashion by using a once-subtracted dispersion relation for the pion charge radius, yielding in the CL

$$r^2_\pi = \frac{6}{\pi} \int_0^\infty \frac{dq^2 \, \Im F_\pi(q^2)}{(q^2)^2} = \frac{N_c}{4\pi^2 (f_\pi^{\rm CL})^2} = \frac{1}{\hat{m}^2} , \tag{22}$$

where we use [5] the GTRs Eq. (2), along with $g = 2\pi/\sqrt{N_c}$ from Eq. (1). This suggests that the tightly bound "fused" $\bar{q}q$ pion charge radius in the CL is

$$r_\pi^{\rm CL} = \frac{1}{\hat{m}} = \frac{197.3 \text{ MeV fm}}{325 \text{ MeV}} \approx 0.61 \text{ fm} , \tag{23}$$

with $\hat{m}_{\rm CL} \approx 325$ MeV $\sim M_N/3$, as expected from the GTR $m_{\rm CL} = f_\pi^{\rm CL} g \approx 90$ MeV $\times 3.628 \approx 325$ MeV.

3 Charged-pion polarizabilities for $\gamma\gamma \to \pi\pi$

For $\gamma\gamma \to \pi\pi$ low-energy scattering, and using units 10^{-42} cm^3 and effective potential $V = -(\alpha_\pi E^2 + \beta_\pi B^2)/2$, Kaloshin *et al.* extracted the observed charged (c) electric and magnetic polarizabilities as

$$(\alpha - \beta)^c = 6.6 \pm 1.2 \ [15] \tag{24}$$

$$(\alpha + \beta)^c = 0.37 \pm 0.08 \ [16,\ 17] \ , \ 0.23 \pm 0.09 \ [16,\ 17] \ , \tag{25}$$

i.e., $\alpha - \beta$ by employing a combined fit to Crystal-Ball [18] and MARK-II [19] data, and $\alpha + \beta$ by fitting CELLO [20] and MARK-II data, respectively. Adding Eqs. (24) and (25) gives

$$\alpha^c = 3.45 \pm 0.60 \ . \tag{26}$$

To compare this "form factor" to theoretical form-factor predictions, we first use $\alpha = e^2/4\pi$ and scale up the potential by 4π. Then α^c in Eq. (26) becomes

$$\alpha_{\pi^+} = (2.75 \pm 0.50) \times 10^{-4} \ {\rm fm}^3 \ . \tag{27}$$

Using the latter scale, the model-independent value is [21]

$$\alpha_{\pi^+} = \frac{\alpha}{8\pi^2 m_\pi f_\pi^2} \, \gamma \ , \tag{28}$$

where $\gamma \equiv F_A(0)/F_V(0)$ is a form-factor ratio found in Sec. 5 to be $2/3$ in the LσM. Thus,

$$\alpha_{\pi^+}^{\rm L\sigma M} = \frac{\alpha}{12\pi^2 m_\pi f_\pi^2} \approx 3.9 \times 10^{-4} \ {\rm fm}^3 \tag{29}$$

is reasonably near the data in Eq. (27) above. It is, moreover, quite close to e.g. the prediction 3.6×10^{-4} fm^3 of a quark confinement model that also yields good results for heavy-meson semi-leptonic form factors [22]. Another consistency check is the detailed quark-plus-meson-loop analysis of Ref. [23]:

$$\alpha_{\pi^+}^{\rm L\sigma M} = \frac{\alpha}{8\pi^2 m_\pi f_\pi^2} - \frac{\alpha}{24\pi^2 m_\pi f_\pi^2} = \frac{\alpha}{12\pi^2 m_\pi f_\pi^2} \ , \tag{30}$$

requiring $\gamma^{\rm L\sigma M} = 2/3$ from Eq. (28).

Finally we comment on low-energy $\gamma\gamma \to 2\pi^0$ scattering, where there is no pole term, and the neutral polarizabilities α_{π^0}, β_{π^0} are much smaller than α_{π^+}, β_{π^+}. In Ref. [24] it was shown that a $\gamma\gamma \to 2\pi^0$ cross section of ~ 10 nb (generated by a $\sigma(700)$ meson pole) reasonably anticipated the later 1990 Crystal-Ball data [18] in the 0.3–0.7 GeV range.

4 Semileptonic weak $K_{\ell 3}$ decay and form-factor scale $f_+(0)$

The semileptonic weak $K^+ \to \pi^0 e^+ \nu$ ($K_{\ell 3}$) decay width is measured as [1]

$$\Gamma(K^+ \to \pi^0 e^+ \nu) \;=\; \frac{\hbar}{\tau_{K^+}}\,(4.87 \pm 0.06)\% \;=\; (25.88 \pm 0.32) \times 10^{-16}\,\mathrm{MeV}\;. \tag{31}$$

Taking a q^2 form-factor dependence $f_+(q^2) = f_+(0)[1 + \lambda_+ q^2/m_\pi^2]$, the standard $V\!-\!A$ (vector here) weak current predicts a $K_{\ell 3}$ decay width ($y = m_{\pi^0}^2/m_{K^+}^2$, $m_e = m_\nu = 0$; see also Ref. [25])

$$\begin{aligned}
\Gamma(K^+ \to\, \pi^0 e^+ \nu) \;&=\; \frac{G_F^2\,|V_{us}|^2\,m_{K^+}^5}{2\,\pi^3\;768}\;f_+^2(0)\Big\{1 - \,8\,y + 8\,y^3 - \,y^4 - \,12\,y^2\,\ln y \\
&\quad + \left(\frac{2}{5}\,\left(1 - \,15\,y - \,80\,y^2 + \,80\,y^3 + 15\,y^4 - \,y^5\right) - \,24\,y^2\,\left(1 + \,y\,\right)\ln y\right) y^{-1}\,\lambda_+ \\
&\quad + \left(\frac{1}{15}\,\left(1 - \,24\,y - \,375\,y^2 + \,375\,y^4 + 24\,y^5 - \,y^6\right) - \,4\,y^2\,\left(3 + \,8\,y + 3\,y^2\right)\ln y\right) y^{-2}\,\lambda_+^2\Big\} \\
&=\; \frac{G_F^2\,|V_{us}|^2\,m_{K^+}^5}{2\,\pi^3\,768}\,f_+^2(0)\Big(0.5792 \;+\; 0.1600\,\frac{m_{K^+}^2}{m_{\pi^0}^2}\,\lambda_+ \;+\; 0.01770\,\frac{m_{K^+}^4}{m_{\pi^0}^4}\,\lambda_+^2\Big) \\
&=\; f_+^2(0)\;(25.90 \pm 0.07) \times 10^{-16}\,\mathrm{MeV}\;,
\end{aligned} \tag{32}$$

where $G_F = 11.6639 \times 10^{-6}$ GeV^{-2}, $V_{us} = 0.2196 \pm 0.0026$, and $\lambda_+ = 0.0278 \pm 0.0019$ [1]. If we neglect here the term quadratic in λ_+, as e.g. done in Ref. [25], the leading factor in Eq. (32) becomes 25.80 instead of 25.90. Moreover, accounting for a nonvanishing electron mass yields a totally negligible correction of the order of 0.001%. In any case, comparison with the data in Eq. (31) clearly shows that the form-factor scale $f_+(0)$ must be near unity. However, electroweak radiative corrections to $\Gamma(K^+ \to \pi^0 e^+ \nu)$ are *not* negligible on the scale of the experimental errors in V_{us} and λ_+, giving rise to an enhancement of $|V_{us}|$ by more than 2% [26], suggesting that $f_+(0)$ should be a trifle less than unity.

As a matter of fact, the nonrenormalization theorem [27] *requires* the form factor $f_+(q^2)$ to be close to unity when $q^2 = 0$. Furthermore, in the infinite-momentum frame (IMF), tadpole graphs are suppressed and so [28]

$$1 - f_+^2(0) \;=\; \mathcal{O}(\delta^2) \;\approx\; 6\% \tag{33}$$

is second order in $SU(3)$-symmetry breaking. Of similar order are, for example, $(m_\pi/m_K)^2 = 7.7\%$, and $(1 - f_K/f_\pi)^2 = 5\%$, for $f_K/f_\pi = 1.22$.

Next we follow the (constituent) quark-model triangle graph of Fig. 2, with

$$\sqrt{2}\,\left\langle \pi^0 | V_\mu^{4-i5} | K^+ \right\rangle \;=\; f_+(t)(p_K + p_\pi)_\mu \;+\; f_-(t)(p_K - p_\pi)_\mu\;. \tag{34}$$

Note that, for this process, the f_- form factor can be disposed of, since it is weighted by $m_e \ll m_K$ [25], giving rise to a m_e^2/m_K^2 suppression of the corresponding contributions to $\Gamma(K^+ \to \pi^0 e^+ \nu)$.

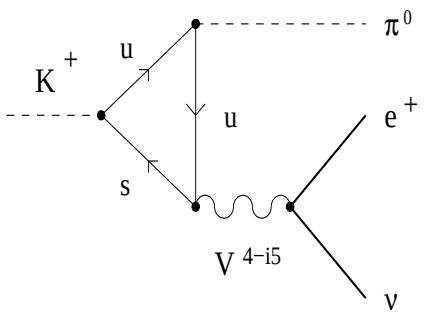


Figure 2: Quark-loop contribution to $K^+ \to \pi^0\, e^+\, \nu$.

To test $SU(2)$-symmetry breaking in $K_{\ell 3}$ decays as in Eqs. (32) and (34) above, we note the present data consistency [1] of $\lambda_+(K^+_{e3}) = 0.0278 \pm 0.0019$, $\lambda_+(K^0_{e3}) = 0.0291 \pm 0.0018$, $\lambda_+(K^+_{\mu 3}) = 0.033 \pm 0.010$, and $\lambda_+(K^0_{\mu 3}) = 0.033 \pm 0.005$. Then, expanding in the $SU(3)$-breaking parameter $\delta = (m_s/\hat{m}) - 1$ (as already used to obtain Eq. (19)) and working in the soft-pion CL, the Feynman graph of Fig. 2 predicts [29] (recall the value of the meson-quark coupling $g \approx 3.628$ in Eq. (1))

$$f_+(0) \;=\; 1 \,-\, \frac{g^2\delta^2}{8\pi^2} \;\approx\; 0.968\ . \tag{35}$$

This value slightly below unity is not only in agreement with the nonrenormalization theorem Eq. (33), as $1 - f_+^2(0) = 1 - (0.968)^2 = 6.3\%$, but also quantitatively compatible with Eqs. (31) and (32), if we account for the mentioned radiative corrections contributing with about -2% to $f_+(0)$, and the experimental errors in V_{us} and λ_+.

5 Semileptonic weak radiative form factors for $\pi^+ \to e^+\nu\gamma$ and $K^+ \to e^+\nu\gamma$

From Ref. [1], the $\pi^+ \to e^+\nu\gamma$ and $K^+ \to e^+\nu\gamma$ matrix elements are

$$M_V \;=\; \frac{-e\, G_F\, V_{qq'}}{\sqrt{2}\, m_P}\, \epsilon^\mu \ell^\nu\, F_V^P\, \epsilon_{\mu\nu\sigma\tau}\, k^\sigma q^\tau\ , \tag{36}$$

$$M_A \;=\; \frac{-ie\, G_F\, V_{qq'}}{\sqrt{2}\, m_P}\, \epsilon^\mu \ell^\nu \{F_A^P\, [(s-t) g_{\mu\nu} - q_\mu k_\nu] + R^P t\, g_{\mu\nu}\}\ , \tag{37}$$

where $V_{qq'}$ is the corresponding Cabibbo-Kobayashi-Maskawa (CKM) mixing-matrix element, ϵ^μ is the photon polarization vector, ℓ^ν is the lepton-neutrino current, q and k are the meson and photon four-momenta, respectively, with $s = q \cdot k$, $t = k^2$, and P stands for π or K. The weak vector (pion) form factor F_V^π in Eq. (36) and the second axial vector form factor R^π in Eq. (37) are model independent [30], with F_V^π determined only by conserved vector currents (CVC), and R^π related via the pion charge radius ($r_\pi = 0.642 \pm 0.002$ fm) to partially conserved (pion) axial

currents (PCAC). Specifically, F_V^π was long ago determined by CVC [30], viz.

$$F_V^\pi(0) \;=\; \frac{\sqrt{2} m_{\pi^+}}{8\pi^2 f_\pi} \;\approx\; 0.027 \;, \tag{38}$$

reasonably close to data [1] 0.017 ± 0.008. Furthermore, PCAC predicts (PCAC is manifest in the LσM [31, 32])

$$R^\pi \;=\; \frac{1}{3} m_{\pi^+} f_{\pi^+} r_{\pi^+}^2 \;=\; 0.064 \pm 0.001 \;, \tag{39}$$

where $f_{\pi^+} = 130.7 \pm 0.1$ MeV [1] and we use $r_{\pi^+} = 0.642 \pm 0.002$ fm. Then Eq. (39) is near data [33] $R^\pi = 0.059^{+0.009}_{-0.008}$.

To apply the LσM theory, we consider the quark-plus-meson-loop graphs of Fig. 3. Then the

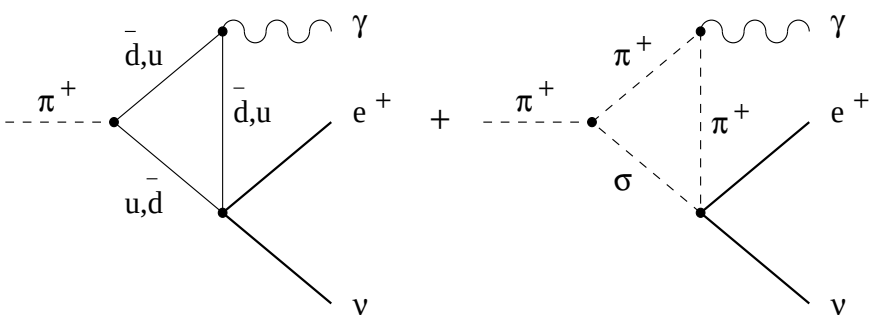


Figure 3: Quark- and meson-loop contribution to $\pi^+ \to \gamma\, e^+ \nu$.

ratio $\gamma = F_A(0)/F_V(0)$ is predicted as [34]

$$\gamma^{\mathrm{L}\sigma\mathrm{M}} \;=\; 1 - \frac{1}{3} \;=\; \frac{2}{3} \;, \tag{40}$$

with

$$F_A^\pi(0) \;=\; \sqrt{2}\, m_\pi [(8\pi^2 f_\pi)^{-1} - (24\pi^2 f_\pi)^{-1}] \;=\; \sqrt{2}\, \frac{m_\pi}{12\pi^2 f_\pi} \;\approx\; 0.0179 \;. \tag{41}$$

Thus, the form-factor ratio of Eq. (41) divided by Eq. (38) gives $\gamma^{\mathrm{L}\sigma\mathrm{M}} = 0.0179/0.027 \approx 0.66$, compatible with Eq. (40) and with data [1]:

$$\gamma_{\mathrm{data}} \;=\; \frac{0.0116 \pm 0.0016}{0.017 \pm 0.008} \;=\; 0.68 \pm 0.33 \;. \tag{42}$$

With hindsight, this ratio $\gamma^{\mathrm{L}\sigma\mathrm{M}} = 2/3$ is near the original current-algebra (CA) estimate 0.6 found in Ref. [35], and exactly the same γ found in Eq. (28) from the LσM Eq. (30).

Extending the above LσM picture to $SU(3)$ symmetry, we first assume a scalar nonet pattern below 1 GeV (e.g. $f_0(600)$, $\kappa(800)$, $f_0(980)$, $a_0(980)$) as found from a kinematic IMF scheme [36], or from a dynamical coupled-channel unitarized model [37]. Then the $K^+ \to e^+ \nu \gamma$ quark-plus-meson LσM form-factor loop amplitudes predict [38] at $k^2 = 0$

$$|F_V^K(0) + F_A^K(0)|_{\mathrm{L}\sigma\mathrm{M}} \;\approx\; 0.109 + 0.044 \;=\; 0.153 \;, \tag{43}$$

close to the $K^+ \to e^+\nu\gamma$ data [1]

$$|F_V^K(0) + F_A^K(0)|_{\rm data} \;=\; 0.148 \pm 0.010 \;. \tag{44}$$

An $SU(3)$ LσM theory is reasonably detailed [39] due to resonances below 1 GeV, but the LσM kaon form-factor sum in Eq. (43) is easily tested via the data in Eq. (44). The same is true for the pion form-factor values in Eqs. (38–42), partly based on the measured pion charge radius [9] $r_\pi = 0.642 \pm 0.002$ fm.

6 Vector-meson dominance: LσM via VPP and VPV or PVV loops

We first confirm the (crucial) value of the pion charge radius [9] $r_\pi = 0.642 \pm 0.002$ fm via Sakurai's vector-meson-dominance (VMD) prediction [40]

$$r_\pi \;=\; \frac{\sqrt{6}}{m_\rho} \;\approx\; 0.63\,{\rm fm}\;. \tag{45}$$

Recall that the tightly bound $\bar{q}q$ chiral pion in Eq. (22), with constituent quark mass $\hat{m} \approx 325$ MeV (near $\hat{m} \approx M_N/3$), has CL charge radius $r_\pi^{\rm CL} = 1/\hat{m} \approx 0.61$ fm. So the close agreement between Eqs. (45) and (22) means we must take the VMD scheme along with the LσM as the basis of our chiral theory.

The ρ^0 form factor predicts, from $udu + dud$ quarks loops in the CL (see Fig. 4),

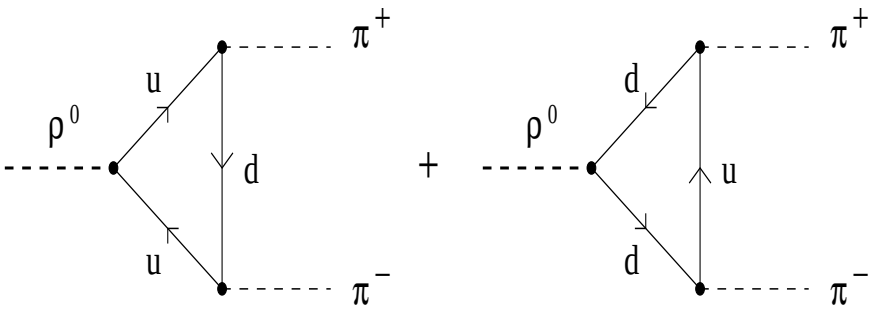


Figure 4: Vector-mesonic VPP quark triangle graphs.

$$g_{\rho\pi\pi} \;=\; -i4N_c\, g^2 g_\rho \int \!d^4p\,(p^2 - \hat{m}^2)^{-2} \;=\; g_\rho\;, \tag{46}$$

by virtue of the LDGE Eq. (13) [11]. Then, folding in the mesonic π-σ-π loop changes the VMD prediction (46) only slightly to [12]

$$g_{\rho\pi\pi} \;=\; g_\rho + \frac{1}{6}\, g_{\rho\pi\pi} \;=\; \frac{6}{5}\, g_\rho\;, \tag{47}$$

compatible with the observed couplings $g_{\rho\pi\pi} \approx 6.04$ and $g_\rho \approx 5.01$, since (for $p_{\rm CM} = 358$ MeV)

$$\Gamma_{\rho\pi\pi} \;=\; \frac{p_{\rm CM}^3\, g_{\rho\pi\pi}^2}{6\pi m_\rho^2} \;=\; 149.2 \pm 0.7\,{\rm MeV} \;\;\Longrightarrow\;\; g_{\rho\pi\pi} \approx 6.04 \tag{48}$$

$$\Gamma_{\rho ee} \;=\; \frac{e^4\, m_\rho}{12\pi g_\rho^2} \;=\; 6.85 \pm 0.11\,{\rm keV} \;\;\Longrightarrow\;\; g_\rho \approx 5.01\ , \tag{49}$$

with $e \approx 0.3028$ (i.e., $\alpha \approx 1/137$). Also, the quark-loop VPV or PVV (see Fig. 5) amplitudes are [41], using $\Gamma_{VPV} = p^3 |F_{VPV}|^2/12\pi$,

$$\begin{aligned} |F(\rho\to\pi\gamma)| \;&=\; \frac{e g_\rho}{8\pi^2 f_\pi} \;\approx\; 0.207\ {\rm GeV}^{-1}\ ,\ \ |F(\omega\to\pi\gamma)| \;=\; \frac{e g_\omega}{8\pi^2 f_\pi} \;\approx\; 0.704\ {\rm GeV}^{-1}\ , \\ |F(\pi^0\to 2\gamma)| \;&=\; \frac{\alpha}{\pi f_\pi} \;=\; \frac{e^2}{4\pi^2 f_\pi} \;\approx\; 0.025\ {\rm GeV}^{-1}\ , \end{aligned} \tag{50}$$

for $g_\rho \approx 5.01$ and $g_\omega \approx 17.06$, very close to the data 0.222 ± 0.012 GeV^{-1} [1], 0.698 ± 0.014 GeV^{-1} [42], 0.0252 ± 0.0009 GeV^{-1} [1], respectively. Equivalently, VMD predicts at tree level $|F_{\rho\pi\gamma}|\, e/g_\rho = |F_{\omega\pi\gamma}|\, e/g_\omega = |F_{\pi^0\gamma\gamma}|/2$, then compatible with the LσM quark loops in Eq. (50).

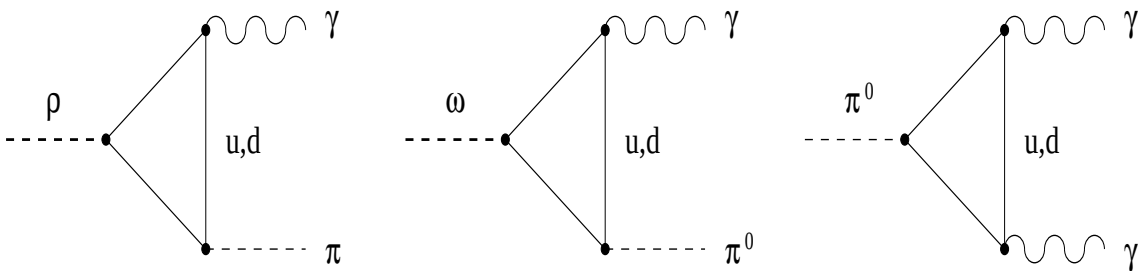


Figure 5: PVV quark triangle graphs for $\rho \to \pi\,\gamma$, $\omega \to \gamma\,\pi^0$, and $\pi^0 \to \gamma\,\gamma$.

7 Nonleptonic weak $K_{2\pi}$ $\Delta I = 1/2$ rule and scalar σ, κ mesons

The well-known [1] $\Delta I = 1/2$ rule $\Gamma(K_S \to \pi^+\pi^-)/\Gamma(K^+ \to \pi^+\pi^0) \approx 450$ for nonleptonic weak $K_{2\pi}$ decays suggests [43] that the parity-violating (PV) amplitude $\langle 2\pi|H_w^{pv}|K_S\rangle$ could be dominated by the $\Delta I = 1/2$ weak transition $\langle\sigma|H_w^{pv}|K_S\rangle$. The σ-pole graph of Fig. 6, with LσM coupling $\langle 2\pi|\sigma\rangle = m_\sigma^2/2f_\pi$ for m_σ near m_K and $\Gamma_\sigma \sim m_\sigma$, predicts [44]

$$|\,\langle 2\pi|H_w^{pv}|K_S\rangle\,| \;=\; \left|\frac{2\,\langle 2\pi|\sigma\rangle\,\langle\sigma|H_w^{pv}|K_S\rangle}{m_K^2 - m_\sigma^2 + i m_\sigma\Gamma_\sigma}\right| \;\approx\; \frac{1}{f_\pi}\,|\langle\sigma|H_w^{pv}|K_S\rangle|\ . \tag{51}$$

But pion PCAC (manifest in the LσM) requires, using the weak chiral commutator $[Q_5 + Q, H_w] = 0$,

$$|\,\langle 2\pi|H_w^{pv}|K_S\rangle\,| \to \frac{1}{f_\pi}\,|\,\langle\pi|[Q_5^\pi, H_w]|K_S\rangle\,| \;\approx\; \frac{1}{f_\pi}\,\left|\left\langle\pi^0|H_w^{pc}|K_L\right\rangle\right|\ , \tag{52}$$

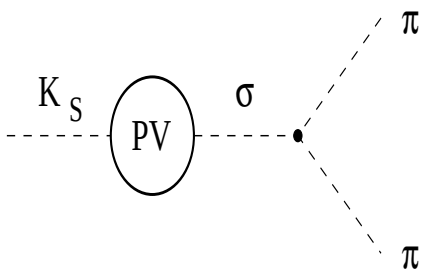


Figure 6: Parity-violating two-pion decay of K_S dominated by σ pole.

with both pions being consistently reduced in Ref. [45]. To reconfirm Eq. (52), one considers the $\Delta I = 1/2$ weak tadpole graph, giving

$$|\langle 2\pi|H_w^{pv}|K_S\rangle| \;=\; \frac{|\langle 0|H_w|K_S\rangle\langle K_S 2\pi|K_S\rangle|}{m_K^2}\;, \tag{53}$$

and then one invokes the Weinberg-Osborn [46] strong chiral coupling $|\langle K_S 2\pi|K_S\rangle| = m_{K_S}^2/2f_\pi^2$, together with the usual PCAC relation $|\langle 0|H_w^{pv}|K_S\rangle| = |2f_\pi\langle\pi^0|H_w^{pc}|K_L\rangle|$, to recover Eq. (52) [47].

In either case, equating Eq. (52) to Eq. (51) leads to

$$|\langle\sigma|H_w^{pv}|K_S\rangle| \;\approx\; \left|\left\langle\pi^0|H_w^{pc}|K_L\right\rangle\right|\;, \tag{54}$$

suggesting that the π and σ mesons are "chiral partners", at least for nonleptonic weak interactions. But of course, Secs. 1–6 above also show that the π and the σ are chiral partners for strong, e.m., and semileptonic weak interactions, as well. To compare this chiral-partner $K \to \pi$ transition with $K_{2\pi}$ data, we return to the PCAC equation (52) to write, for $f_\pi \approx 93$ MeV,

$$|\langle 2\pi|H_w^{pv}|K_S\rangle| \;\approx\; \frac{1}{f_\pi}\left|\left\langle\pi^0|H_w^{pc}|K_L\right\rangle\right| \;\approx\; 38\times 10^{-8}\;\text{GeV}\;, \tag{55}$$

midway between the observed $K_S \to \pi^+\pi^-$ and $K_S \to \pi^0\pi^0$ amplitudes

$$\left|M^{+-}_{K_S\to\pi\pi}\right|_{\rm PDG} \;=\; m_{K_S}\left[\frac{8\pi\Gamma^{K_S}_{+-}}{q}\right]^{\frac{1}{2}} \;=\; (39.1\pm 0.1)\times 10^{-8}\;\text{GeV}\;, \tag{56}$$

$$\left|M^{00}_{K_S\to\pi\pi}\right|_{\rm PDG} \;=\; m_{K_S}\left[\frac{16\pi\Gamma^{K_S}_{00}}{q}\right]^{\frac{1}{2}} \;=\; (37.1\pm 0.2)\times 10^{-8}\;\text{GeV}\;, \tag{57}$$

suggesting $|\langle\pi^0|H_w^{pc}|K_L\rangle| \approx 3.58\times 10^{-8}$ GeV^2. In fact, when one statistically averages *eleven* first-order weak data sets for $K_S \to 2\pi$, $K \to 3\pi$, $K_L \to 2\gamma$, $K_L \to \mu^+\mu^-$, $K^+ \to \pi^+e^+e^-$, $K^+ \to \pi^+\mu^+\mu^-$, and $\Omega^- \to \Xi^0\pi^-$, one finds [48]

$$\left|\left\langle\pi^0|H_w^{pc}|K_L\right\rangle\right| \;=\; \left|\left\langle\pi^+|H_w^{pc}|K^+\right\rangle\right| \;=\; (3.59\pm 0.05)\times 10^{-8}\;\text{GeV}^2\;. \tag{58}$$

To induce theoretically at the quark level the ΔI=1/2 $s \to d$ single-quark-line (SQL) transition scale β_w in a model-independent manner, one considers the second-order weak (see Fig. 7) K_L–K_S

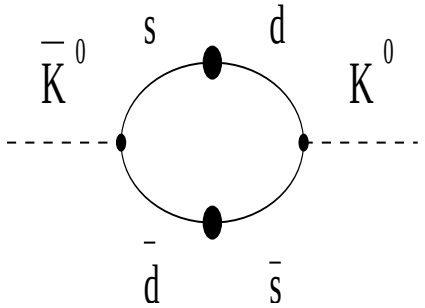


Figure 7: $\bar{K}^0 \leftrightarrow K^0$ SQL graph. Each dot represents the SQL weak scale β_w.

mass difference Δm_{LS} diagonalized to [49]

$$2\beta_w^2 \;=\; \frac{\Delta m_{LS}}{m_K} \;=\; (0.70126 \pm 0.00121) \times 10^{-14} \;\;\Longrightarrow\;\; |\beta_w| \;\approx\; (5.9214 \pm 0.0051) \times 10^{-8}\;. \quad (59)$$

Then using Eq. (58), one predicts from the soft-meson theorem, or from Cronin's chiral Lagrangian [50],

$$\left|\left\langle \pi^0 | H_w^{pc} | K_L \right\rangle\right| \;=\; 2\beta_w m_{K_L}^2 \frac{f_K}{f_\pi} \;=\; (3.5785 \pm 0.0031) \times 10^{-8}\,\mathrm{GeV}^2\;, \quad (60)$$

given $f_K/f_\pi \approx 1.22$. This SQL scale β_w in Eq. (59) and the $K \to \pi$ weak amplitude in Eq. (60) (or in Eq. (58)), correspond to a "truly weak" interaction, which Weinberg [51] shows cannot be transformed away in the electroweak standard model.

To test the latter weak scale (60) (or the similar data averages (58), we first re-express the neutral chiral-partner relation (extended to the κ transition [44]) as

$$\left\langle \pi^0 | H_w^{pc} | K^0 \right\rangle \;=\; \left\langle \sigma | H_w^{pv} | K^0 \right\rangle \;=\; \left\langle \pi^0 | H_w^{pv} | \kappa^0 \right\rangle \;=\; \frac{1}{\sqrt{2}}\, 3.58 \times 10^{-8}\,\mathrm{GeV}^2 \;=\; 2.53 \times 10^{-8}\,\mathrm{GeV}^2\;. \quad (61)$$

We fix this $\kappa^0 \to \pi^0$ weak transition (61) to the weak PV K^0 tadpole graph of Fig. 8, via the $K^0 \to$ vacuum PCAC scale, as

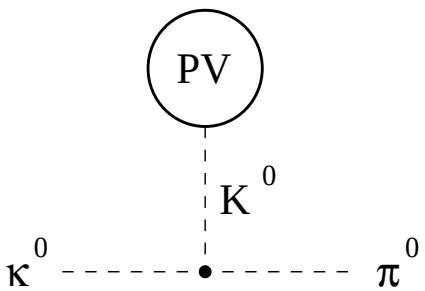


Figure 8: Parity-violating weak K^0 tadpole graph for $\kappa^0 \to \pi^0$ transition.

$$\left|\left\langle 0 | H_w^{pv} | K^0 \right\rangle\right| \;=\; \frac{2f_\pi^2}{1 - m_\pi^2/m_K^2} \left|\left\langle 2\pi^0 | H_w^{pv} | K^0 \right\rangle\right| \;=\; 0.51 \times 10^{-8}\,\mathrm{GeV}^3\;, \quad (62)$$

using $|\langle 2\pi^0|H_w^{pv}|K^0\rangle| = 26.26\times 10^{-8}$ GeV from data, while eliminating the 4% $\Delta I = 3/2$ component (see Ref. [51], third paper). Then Fig. 8 predicts the amplitude magnitude

$$\left|\left\langle \pi^0|H_w^{pv}|\kappa^0\right\rangle\right| \;=\; \frac{|\langle 0|H_w^{pv}|K^0\rangle|}{m_{K^0}^2}\, g_{\kappa^0 K^0\pi^0} \;\approx\; 2.53\times 10^{-8}\,\mathrm{GeV}^2\;, \tag{63}$$

scaled to Eq. (61) above, *provided* one uses the LσM coupling, for $f_\pi = 92.4$ MeV [1],

$$|g_{\kappa^0 K^0\pi^0}| \;=\; \frac{m_\kappa^2 - m_K^2}{4f_\pi} \;=\; 1.229\ \mathrm{GeV}\;, \tag{64}$$

corresponding to a κ mass of 838 MeV. This value is not too distant from our earlier $m_\kappa = 730$–800 MeV predictions [36, 37], and the very recent E791 observed mass $m_\kappa \approx 800$ MeV [52]. Moreover, the $SU(3)$ analogue to Eq. (64), i.e., $|g_{\sigma\pi\pi}| = (m_\sigma^2 - m_\pi^2)/2f_\pi$ suggests $m_\sigma = 687$ MeV, reasonably near the predicted CL-LσM value [5, 53] $m_\sigma = 650$ MeV.

8 Summary and conclusions

In Sec. 1 we reviewed the solution of the LσM at the quark-loop level. In Sec. 2 we used $SU(2)$, $SU(3)$ Goldberger–Treiman quark relations to normalize the π and K form factors to unity at $k^2 = 0$, after which we differentiated these form factors to predict the LσM charge radii, both being compatible with data. Next in Sec. 3 we briefly reviewed e.m. charged-pion polarizabilities for $\gamma\gamma \to \pi\pi$, and compared them with LσM predictions. In Sec. 4 we used quark loops to match the observed form factor $f_+(0)$. In Sec. 5 we showed that the LσM form factors F_V^π, R^π, $F_V^K + F_A^K$, and the ratio F_A^π/F_V^π are all in agreement with the measured values. Then in Sec. 6 we compared tree-level VMD with LσM VPP and PVV quark loops. Both theories agree well with data. Finally, in Sec. 7 we successfully extended this LσM picture to nonleptonic weak decays, in particular to the $\Delta I = 1/2$-dominated $K_{2\pi}$ decays and inferred $\sigma(687)$ and $\kappa(838)$ masses. All our main results are summarized in Table 1, in confrontation with experiment.

Next, we discuss low-energy QCD. While an exact match via the LσM is not possible, QCD at the 1-GeV scale generates a dynamical quark mass [54] $m_{\rm dyn} = [4\pi\alpha_s \left\langle -\bar\psi\psi\right\rangle_{1\ \mathrm{GeV}}/3]^{1/3} \approx 320$ MeV, near the LσM quark mass $\hat m = 2\pi f_\pi/\sqrt{3} \approx 325$ MeV in the CL. Such approximate agreement also holds for the quark condensate as well. Moreover, the frozen coupling strength in QCD at infrared freeze-out [55] is $\alpha_s = \pi/4$, with $\alpha_s^{\rm eff} = (4/3)\alpha_s = \pi/3$. This exactly matches the LσM strength $\alpha_{\rm L\sigma M} = g^2/4\pi = \pi/3$, with $g = 2\pi/\sqrt{3}$. Also, QCD with $\alpha_s(m_\sigma) = \pi/4$ leads to [56] $m_\sigma^2/m_{\rm dyn}^2 = \pi/\alpha_s(m_\sigma) \approx 4$, simulating the NJL–L$\sigma$M value $m_\sigma^2/\hat m^2 = 4$ in the CL. Lastly, the chiral restoration temperature T_c computed in $N_f = 2$ lattice simulations gives [57] $T_c = 173 \pm 8$ MeV, close to the LσM value [58] $T_c = 2f_\pi \approx 180$ MeV in the CL.

	LσM (CL)	experiment
$\langle r_{\pi^+} \rangle$	0.61 fm	(0.642$\pm$0.002) fm [9]
$\langle r_{K^+} \rangle$	0.49 fm	(0.560$\pm$0.031) fm [1]
α_{π^+}	3.9×10^{-4} fm^3	$(2.75\pm0.50)\times10^{-4}$ fm^3 (see Sec. 3)
$f_+(0)$	0.968	(see discussion in Sec. 4)
R^π	0.064$\pm$0.001	$0.059^{+0.009}_{-0.008}$ [33]
F_A^π	0.0179	0.0116$\pm$0.0016 [1]
F_V^π	0.027	0.017$\pm$0.008 [1]
$\gamma^\pi = \frac{F_A^\pi(0)}{F_V^\pi(0)}$	$\frac{2}{3}$	0.68$\pm$0.33 [1]
$\left\|F_V^K(0) + F_A^K(0)\right\|$	0.153	0.148$\pm$0.010 [1]
$\|F(\rho \to \pi\gamma)\|$	0.207 GeV^{-1}	(0.222$\pm$0.012) GeV^{-1} [1]
$\|F(\omega \to \pi\gamma)\|$	0.704 GeV^{-1}	(0.698$\pm$0.014) GeV^{-1} [42]
$\|F(\pi^0 \to \gamma\gamma)\|$	0.025 GeV^{-1}	(0.0252$\pm$0.0009) GeV^{-1} [1]
$\|\langle 2\pi\|H_w^{pv}\|K_S\rangle\|$	38×10^{-8} GeV	$(39.1 \pm 0.1) \times 10^{-8}$ GeV $(\pi^+\pi^-)$ $(37.1 \pm 0.2) \times 10^{-8}$ GeV $(\pi^0\pi^0)$ } [1]
m_κ	838 MeV	797$\pm$19$\pm$43 MeV [52]

Table 1: Confrontation of the LσM in the chiral limit with experiment.

To conclude, we mention a very recent large-N_c renormalization-group-flow analysis of the quark-level LσM [59], using the Schwinger proper-time regularization, which finds (for $f_\pi = 93$ MeV) $\lambda = 23.6$, $g = 3.44$, $m_q = 320$ MeV, and $m_\sigma = 650$ MeV, strikingly close to our above theoretical values $\lambda = 8\pi^2/3 = 26.3$, $g = 2\pi/\sqrt{3} \approx 3.628$, $m_q = 325$ MeV, and $m_\sigma = 650$ MeV, respectively. Therefore, our present results, as well as our recent findings in Ref. [4], appear to confirm the assumption of the authors of Ref. [59]: *"We assume the linear σ model to be a valid description of Nature below scales of 1.5 GeV."*

Acknowledgments.

The authors are indebted to A. E. Kaloshin for valuable information on pion polarizabilities. This work was partly supported by the *Fundação para a Ciência e a Tecnologia* (FCT) of the *Ministério do Ensino Superior, Ciência e Tecnologia* of Portugal, under Grant no. PRAXIS XXI/-BPD/20186/99 and under contract number CERN/P/FIS/43697/2001.

APPENDIX

A Tree-level LσM

From the $SU(2)$ LσM of Ref. [60] one knows the interacting Lagrangian density relative to the true vacuum [5]:

$$\mathcal{L}^{\rm int}_{\rm L\sigma M} = g\,\bar\psi(\sigma + i\gamma_5\vec\tau\cdot\vec\pi)\psi \;+\; g'\,\sigma\,(\sigma^2+\pi^2) \;-\; \frac{\lambda}{4}\,(\sigma^2+\pi^2)^2 \;-\; f_\pi g\,\bar\psi\psi\ . \tag{65}$$

A tree-level theory then implies the chiral relations in the CL [31, 32], with constituent quark mass m_q,

$$g \;=\; \frac{m_q}{f_\pi} \quad , \quad g' \;=\; \frac{m_\sigma^2}{2f_\pi} \;=\; \lambda\, f_\pi\ . \tag{66}$$

B Bootstrapping $g_{\sigma\pi\pi} \to g'$ and $\lambda_{\rm box} \to \lambda_{\rm tree}$

The $\sigma\pi\pi$ or $\sigma\sigma\sigma$ u,d quark triangle graphs [5, 60] induced by $\mathcal{L}^{\rm int}_{\rm L\sigma M}$ in Eq. (65) implies in the CL

$$g_{\sigma\pi\pi} \;=\; -8ig^3N_cm_q\int \bar d^4p\,\left[p^2-\hat m^2\right]^{-2} \;=\; 2gm_q\ , \tag{67}$$

due to the LDGE (13). Then the GTR Eq. (2), together with $m_\sigma = 2m_q$, reduces Eq. (67) to

$$g_{\sigma\pi\pi} \;=\; 2gm_q \;=\; \frac{m_\sigma^2}{2f_\pi} \;=\; g'\ , \tag{68}$$

the tree-level cubic meson LσM coupling in Eq. (66). Also the $\pi\pi\pi\pi$ (or $\sigma\sigma\sigma\sigma$, $\pi\pi\sigma\sigma$) quark box graph [5, 60] generates in the CL

$$\lambda_{\rm box} \;=\; -8ig^4N_c\int \bar d^4p\,\left[p^2-\hat m^2\right]^{-2} \;=\; 2g^2 \;=\; \frac{g'}{f_\pi} \;=\; \lambda_{\rm tree}\ , \tag{69}$$

again due to the LDGE (13). Note that the cubic and quartic LσM tree couplings in Eq. (66) are dynamically loop-generated in Eqs. (67) and (69), respectively. Both are analytic, nonperturbative bootstrap procedures [5].

C Dim-reg. lemma generating quark and σ mass

The Nambu $\delta m_q = m_q$ (constituent-) quark mass-gap tadpole graph [5, 60] generates quark mass. However, this quadratically divergent term, subtracted from the LDGE (13), in fact scales to quark mass *independently* of quadratically divergent terms, by virtue of the dimensional-regularization (dim-reg.) lemma [5]

$$I \;=\; \int \bar d^4p\,\left[\frac{m^2}{(p^2-m^2)^2} \;-\; \frac{1}{p^2-m^2}\right] \;=\; \lim_{\ell\to 2}\frac{im^{2\ell-2}}{(4\pi)^\ell}\,[\Gamma(2-\ell)+\Gamma(1-\ell)] \;=\; -im^2(4\pi)^{-2}\ , \tag{70}$$

due to the gamma-function *identity* $\Gamma(2-\ell)+\Gamma(1-\ell)=\Gamma(3-\ell)/(1-\ell)\to -1$ as $\ell\to 2$. To reconfirm this dim-reg.-lemma "trick" (70), we invoke the partial-fraction *identity*

$$\frac{m^2}{(p^2-m^2)^2}\ -\ \frac{1}{p^2-m^2}\ =\ \frac{1}{p^2}\left[\frac{m^4}{(p^2-m^2)^2}\ -\ 1\right]\ , \tag{71}$$

integrated via $\int \not{d}^4p$ as in the I integral on the l.h.s. of Eq. (70). Then dropping the massless-tadpole integral $\int \not{d}^4p/p^2=0$ (as done in dimensional, analytic, zeta-function, and Pauli–Villars regularizations [5, 61]), and Wick rotating $d^4p=i\pi^2p_{\rm E}^2dp_{\rm E}^2$, the Euclidean integral becomes

$$I\ =\ -\frac{im^4}{(4\pi)^2}\int_0^\infty\frac{dp_{\rm E}^2}{(p_{\rm E}^2+m^2)^2}\ =\ -\frac{im^2}{(4\pi)^2}\ , \tag{72}$$

identical to the r.h.s. of Eq. (70).

In order to further justify the neglect of $\int \not{d}^4p/p^2$, we invoke the Karlson trick [62] (long advocated by Schwinger)

$$\frac{d}{dm^2}\int\frac{d^4p}{p^2-m^2}\ =\ \int\frac{d^4p}{(p^2-m^2)^2}\ , \tag{73}$$

and compute [63]

$$(2\pi)^4\frac{dI}{dm^2}\ =\ \int\frac{d^4p}{(p^2-m^2)^2}\ +\ 2m^2\int\frac{d^4p}{(p^2-m^2)^3}\ -\ \frac{d}{dm^2}\int\frac{d^4p}{p^2-m^2}\ , \tag{74}$$

with the first and third terms cancelling due to Eq. (73). Then the remaining, finite second term in Eq. (74) gives

$$(2\pi)^4\frac{dI}{dm^2}\ =\ 2m^2\left(-\frac{i\pi^2}{2m^2}\right)\ =\ -i\pi^2\ , \tag{75}$$

which is the *same* result as differentiating the dim-reg. lemma (70):

$$(2\pi)^4\frac{dI}{dm^2}\ =\ (-i\pi^2)\frac{dm^2}{dm^2}\ =\ -i\pi^2\ . \tag{76}$$

So far we have only assumed $\int \not{d}^4p/p^2$ is independent of m^2, so that $(d/dm^2)\int \not{d}^4p/p^2=0$.

But to demonstrate that the $\int dm^2$ integration constant *vanishes,* i.e., $\int d^4p/p^2=\Lambda^2=0$, we invoke the implied dimensional-analysis relations

$$\int\frac{d^4p}{p^2}\ =\ 0\ ,\ \int\frac{d^4p}{p^2-m^2}\ \propto\ m^2\ ,\ \int\frac{d^4p}{p^2-m_\sigma^2}\ \propto\ m_\sigma^2 \tag{77}$$

to solve B. W. Lee's null-tadpole sum [64], which characterizes the true vacuum for $N_f=2$ as [5]

$$(2m_q)^4N_c\ =\ 3m_\sigma^4 \tag{78}$$

(with the factor of 3 due to σ-σ-σ combinatorics) in the CL $m_\pi=0$, meaning $N_c=3$ when $m_\sigma=2m_q$.

Thus, $\int d^4p/p^2$ indeed vanishes as suggested [5, 61]. Then appendices A, B, and C loop-generate Eq. (1) via the LDGE Eq. (13) and the dim.-reg. lemma Eq. (70) [5, 60] .

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