Fascinating Field Theory: Quantum Field Theory, Renormalization Group and Lattice Regularization *

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Contents

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1 Introduction

In this introduction we try to explain and motivate some of the notions entering the title of these lectures.

1.1 Why Field Theory (as opposed to point mechanics)?

If one is dealing with a large number of degrees of freedom, interesting collective phenomena arise.

Consider e.g. the pendulum. Its motion is a simple problem in classical mechanics. If we consider many of them coupled by strings, modes might occur which involve the coordinated, correlated motion of all the degrees of freedom (*waves*). For these kinds of phenomena a field theoretic description is very natural.

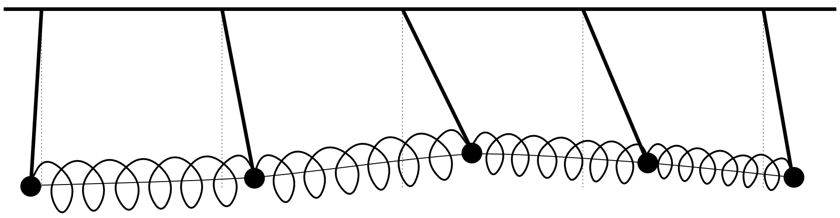

Figure 1: Collective phenomena.

1.2 Why Quantum Field Theory (QFT) (rather than Quantum Mechanics (QM) for the collective excitations)?

In microphysics, the birth and death, the *creation* and *annihilation* of excitations (particles) is a basic feature, which is observed everywhere.

Take for example the absorption and emission of a photon by an atom. QM already has problems with this relatively simple example, especially with induced emission.[1] There are, however, high energy scattering events where the number of produced particles lies in the hundreds, even in the thousands. QM works

[1] Strictly speaking, it does not belong to the realm of QM.

with normalized wavefunctions and the associated interpretation excludes the possibility of particle creation and annihilation.

We need a prescription with an infinite number of degrees of freedom. This is just what QFT provides us.

1.3 Why Lattice Regularization?

Some of the mathematical operations which enter the definition of a QFT require a careful limiting procedure. Take the derivative of a function for example

$$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}. \tag{1}$$

How to introduce and how to remove the *regularization* in a QFT - this is a highly non-trivial problem.

Accepting the fact that we need some regularization scheme – why should we resort to lattice regularization? Let us mention two of the reasons:

- In general, in any problems which require numerical analysis (e.g. integrals, partial differential equations, ...), the standard procedure is to introduce meshes. Also the path integral is defined in a natural way via meshes. Consider the probability amplitude for the propagation of a particle from the coordinates (x_i, t_i) to (x_f, t_f) in QM:

  $$K(x_f, t_f; x_i, t_i) = \sum_{\text{all paths}} e^{iS_{cl}(x(t))}, \tag{2}$$

  where S_{cl} is the classical action and the sum extends over all paths connecting the points (x_i, t_i) and (x_f, t_f). The sum over all paths in equation (2) is defined by introducing a space-time lattice (see figure 2)).

- Other regularization schemes have shortcomings. For example the dimensional regularization can only be applied in the context of perturbation theory.

  Lattice regularization is very convenient for gauge systems, or for theories with constrained variables. Choosing a momentum cutoff, the constraints may become very cumbersome. Consider for example the non-linear σ-model, where the spins lie on a unit sphere: $\vec{S}(\vec{x})^2 = 1$. This constraint is simple in configuration space, but complicated in momentum space. In addition to that, a momentum cutoff violates gauge invariance.

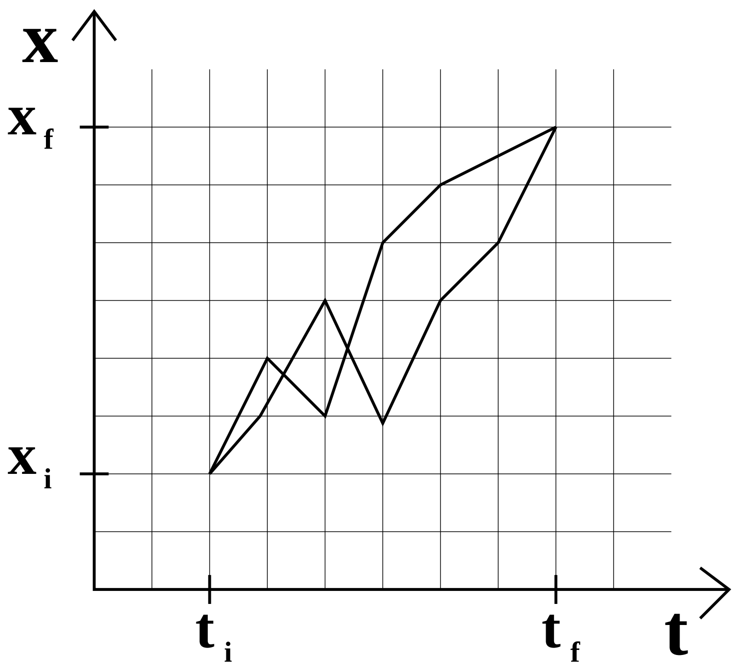

Figure 2: Paths on a space-time lattice.

1.4 Why Renormalization Group?

We begin with a slight detour, asking ourselves about the common feature of all Field Theories (QCD, QED, electro-weak interaction, Φ^4, ...). Their most important common property is *locality*. Locality means that the elementary interaction in the action takes place only between infinitesimally separated points, like in the expression

$$\frac{\Phi(x+a)-\Phi(x)}{a}, \tag{3}$$

where the separation a approaches zero in the limiting procedure.

On the other hand, we are interested in finite correlation lengths of the systems described by Field Theories. In QCD, for example, we want to obtain a correlation length of the order fm, which is the typical scale for hadronic objects. Therefore some miraculous thing has to happen, which ensures that we obtain some finite correlation length at the end in the solution (in the Greens functions), despite the fact that the distance of our basic interaction in the action reaches zero. Collective behavior is needed to overcome this paradox.

Let us look at the relevant scales in momentum space (fig. 3). We are interested

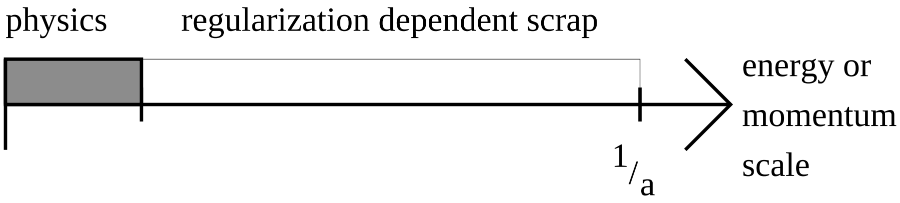

Figure 3: Momentum scales.

in some region of physical momenta, which is well below the cutoff. Nobody can tell how the elementary interaction between, say, an electron and a photon at the cut-off should look like. Therefore the results of the theory in the physical region should be independent of the details of the interaction near the cutoff (i.e. independent of the details of the interaction, or of the regularization scheme). This property, called universality, is produced typically by the collective motion of a large number of degrees of freedom. This is the property which allows us to calculate the anomalous magnetic moment of the electron to many digits precision in spite of our limited knowledge on the form of the interaction at very high energies.

It would be useful to define a procedure, which eliminates the uninteresting degrees of freedom at high momenta, but takes into account their effect on the remaining variables exactly. This procedure is the *Renormalization Group Transformation (RG)*. The RG provides qualitative results on universality and renormalizibility of a theory, but also quantitative results, as the calculation of critical exponents by applying Wilson's ϵ-expansion (see [Wil74]).

2 Quantum Field Theories (QFTs) versus critical phenomena in classical statistical physics

2.1 Formal relation between QFTs and classical statistical physics

Consider an n-point function in QFT in Minkowski space, with $\Phi(x)$ as the generic field

$$\langle\, 0 \mid \mathrm{T}\left(\hat{\Phi}(x_1), \hat{\Phi}(x_2), \ldots, \hat{\Phi}(x_n)\right) \mid 0\,\rangle. \tag{4}$$

In path integral language this expression is equal to

$$= \frac{\int \mathrm{D}\Phi\, \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n) \exp\left(i \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\vec{x}\, \mathcal{L}(\Phi, \partial_\mu \Phi)\right)}{Z} \quad , \tag{5}$$

where $Z = \int \mathrm{D}\Phi \exp\left(i \int dt \int d\vec{x}\, \mathcal{L}\right)$ is the normalization and $x = (t, \vec{x}) = (t, x_1, \ldots, x_s)$.

For the Lagrangian $\mathcal{L}$ we might take

$$\mathcal{L}_{\text{Minkowski}} = \frac{1}{2}\partial_0\partial_0 - \sum_{i=1}^{d_s} \frac{1}{2}\partial_i\partial_i - V(\Phi) \quad . \tag{6}$$

For many reasons it is helpful to perform an analytic continuation and go to Euclidean space,

$$\begin{aligned} t = x_0 \quad &\to \quad -ix_d \\ i\int dt \int d\vec{x} \quad &\to \quad \int d^d x \\ \text{where} \quad & d = d_s + 1 \end{aligned} \tag{7}$$

By Wick rotation, the relative minus sign between the time and the spatial derivatives disappears,

$$\mathcal{L}_{\text{Minkowski}} \to -\left(\sum_{i=1}^{d} \frac{1}{2}\partial_i\partial_i + V(\Phi)\right) = -\mathcal{E}(\Phi, \partial\Phi) \tag{8}$$

and we get

$$\begin{aligned} Z \quad &= \quad \int \mathrm{D}\Phi \exp\left(-\int d^d x\, \mathcal{E}(\Phi, \partial\Phi)\right) \\ \langle \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)\rangle \quad &= \quad \frac{\int \mathrm{D}\Phi\, \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)\, \exp\left(-\int d^d x\, \mathcal{E}\right)}{Z} . \end{aligned} \tag{9}$$

Equation (9) gives nothing but the partition function and correlation functions for a system in classical statistical mechanics (with a trivial rescaling of Φ, we can obtain $1/T$ in the Boltzmann factor, if needed).

2.2 Mass (gap) m_{df}, correlation length ξ_{df}

The subscript *df* denotes *dimensionful quantities*. The n-point (correlation) functions also determine the spectrum of the theory. Take (in Minkowski space) the operator in the Heisenberg picture

$$\hat{\Phi}(t,\vec{0}) = e^{i\mathcal{H}t}\hat{\Phi}_S e^{-i\mathcal{H}t}, \tag{10}$$

where $\hat{\Phi}_S$ is the field operator in the Schrödinger picture at the space point $\vec{0}$.

Then

$$\langle\, 0 \mid \hat{\Phi}(t,\vec{0})\, \hat{\Phi}(0,\vec{0}) \mid 0\,\rangle = \sum_n e^{-i(E_n-E_0)t} \left| \langle\, 0 \mid \hat{\Phi}_S \mid n\,\rangle \right|^2 , \tag{11}$$

where we have used the closure relation

$$1 = \sum_n \mid n\,\rangle\langle\, n \mid \tag{12}$$

for the energy eigenstates $\mid n\,\rangle$ of the system. If we perform the analytic continuation to Euclidean space, we obtain

$$t \;\; \to \;\; -ix_d$$

$$\langle\, 0 \mid \hat{\Phi}(t,\vec{0})\hat{\Phi}(0,\vec{0}) \mid 0\,\rangle \;\; \to \;\; \sum_n e^{-(E_n-E_0)x_d} \left| c_n \right|^2 . \tag{13}$$

If the state $\hat{\Phi}_S \mid 0\,\rangle$ is orthogonal to the groundstate $\mid 0\,\rangle$ then

$$\langle\, 0 \mid \hat{\Phi}(t,\vec{0})\, \hat{\Phi}(t,\vec{0}) \mid 0\,\rangle \;\; \stackrel{x_d\to\infty}{\longrightarrow} \;\; e^{-\Delta E x_d} \left| c_1 \right|^2 ,$$

$$\Delta E = E_1 - E_0 \doteq m_{df}. \tag{14}$$

The dimensionful quantity m_{df} is the *mass gap* in the channel of $\hat{\Phi}$. As m_{df} determines how fast the correlator in equation (14) falls off as a function of x_d, the inverse of m_{df} can be identified with the correlation length ξ_{df}:

$$\xi_{df} = \frac{1}{m_{df}}. \tag{15}$$

Note again that m_{df} and ξ_{df} are dimensionful physical observables and their physical units may be given for example by: $[m_{df}] = g$, $[\xi_{df}] = cm$.

2.3 QFTs versus critical phenomena

There is a deep similarity between the problems of QFT and those of critical phenomena. For illustration, consider a large piece of magnet. The magnet is built of atoms forming a lattice structure with a lattice distance $a \sim$ O(Å).

At a given generic temperature $T > T_C$, the distance over which the magnetic spins are correlated (correlation length) is of the order of the lattice spacing (see fig. 4 on the next page). Let us tune the temperature towards the critical temperature where spontaneous magnetization sets in. In approaching T_C, the correlation length gets larger and larger, several hundreds of lattice units (hundreds of Ås), and beyond. A field theorist looks at this problem through his special glasses. These are reducing glasses. Through his glasses, the original lattice unit and correlation length is $\mathrm{O}(1fm)$. This is a trivial rescaling. The most important thing is that the field theorist increases the reducing factor of his glasses in such a way that through his glasses, the correlation length stays the order of $1fm$ even when the temperature gets close to the Curie point (see fig. 4). For the solid state physicist, the correlation length approaches macroscopical distances, while looking through the increasingly reducing glasses of the field theorist, the correlation length remains the same, but the lattice becomes increasingly fine grained. At the Curie point, the field theorist obtains a continuum field theory ($a \to 0$) with finite $\mathrm{O}(1fm)$ correlation length. His colleague has a lattice with fixed lattice unit ($a \sim 1$ Å) and macroscopical correlation length. It is clear, however, that the main difficulty – obtaining and controlling the large scale collective behavior – is a common problem for them.

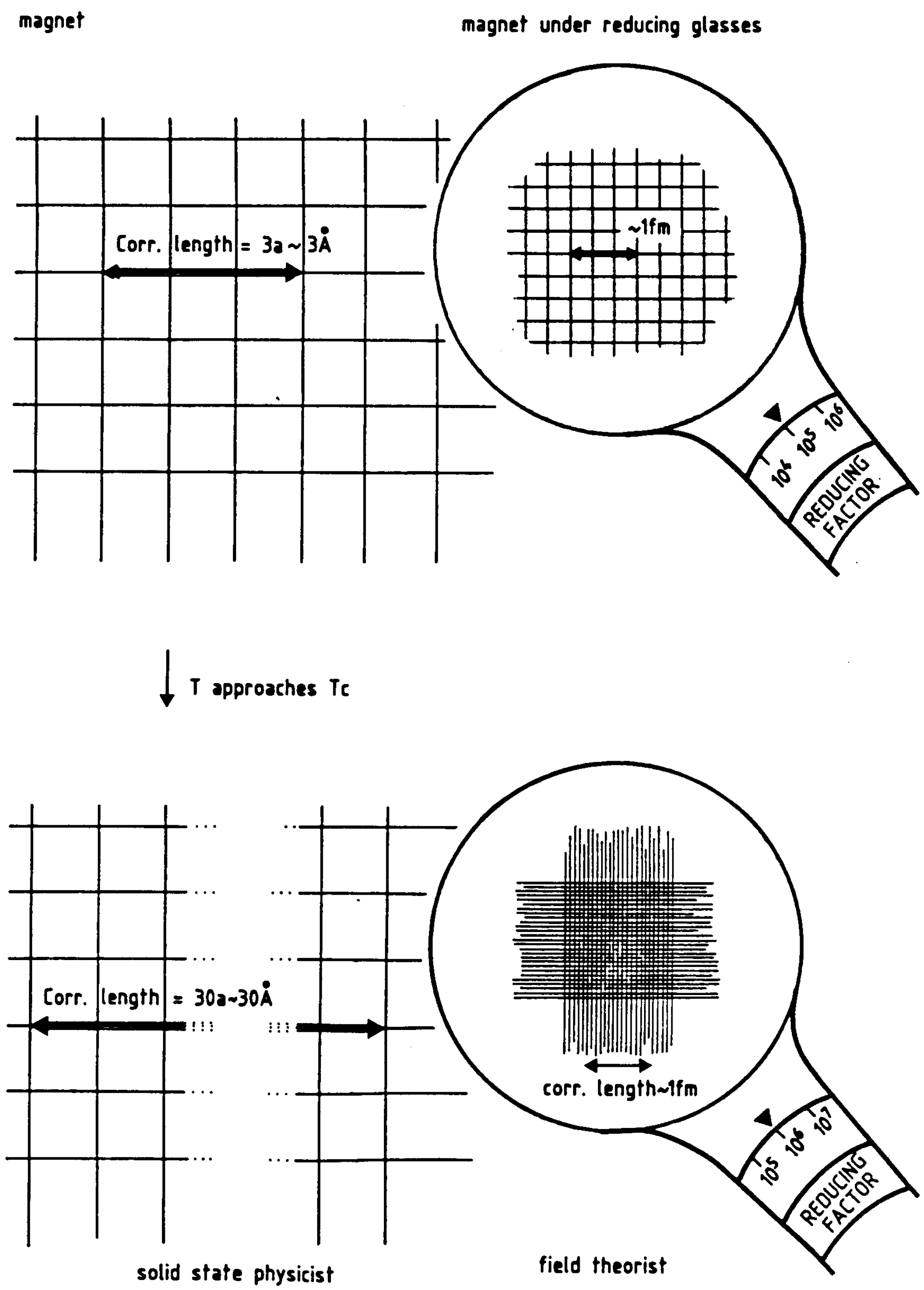

Figure 4: Solid State Physics versus Field Theory.

3 Renormalization Group (RG) transformation

Let us start to define RG transformations in momentum space first. The systematic elimination of short distance (large momentum) fluctuations is done by:

- *thinning out* the variables
- *relabelling* and *rescaling* the remaining variables

In calculations $\Phi(x)$ is written for a scalar field, $\tilde{\Phi}(k)$ for its Fourier transform:

$$\Phi(x) = \int_k e^{-ikx}\,\tilde{\Phi}(k) \quad ,$$

using the following abbreviation for the d–dimensional integration in Euclidean momentum space:

$$\int_k = \int_{-\infty}^{\infty} \frac{d^d k}{(2\pi)^d} \quad .$$

3.1 RG transformation in momentum space

We choose a cutoff in momentum space:

$$|k| \leq \Lambda^{cut} \quad .$$

The next step is to express everything in terms of *dimensionless* quantities in units of the cutoff, i.e. introduce the dimensionless momentum variable q :

$$q_\mu = \frac{k_\mu}{\Lambda^{cut}} \quad \Rightarrow \quad |q| \leq 1 \quad ,$$

and the dimensionless field variables $\varphi(q)$:

$$\varphi(q) = \tilde{\Phi}(k) \cdot \left(\Lambda^{cut}\right)^{\frac{d+2}{2}} \quad .$$

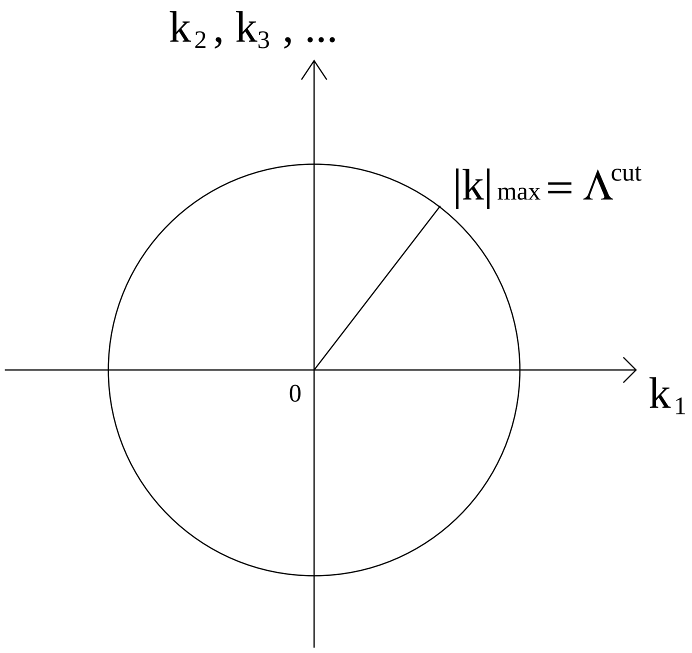

Figure 5: Considered range of momentum.

The generating functional in Euclidean space then reads

$$Z = \prod_{|q|\leq 1} \int d\varphi\,(q)\, e^{-S(\varphi)} \quad . \tag{16}$$

The idea is now to integrate out in a cascading procedure the irrelevant high momentum parts of the degrees of freedom in Z to come down to relevant momentum scales which determine physics. The cascading procedure is done as follows. One splits up the fields $\varphi\,(q)$ into two parts, $\varphi_0(q)$, which is defined in the region $|q| \leq \frac{1}{2}$, and $\varphi_1(q)$, which is defined in the region $\frac{1}{2} \leq |q| \leq 1$:

$$\varphi\,(q) = \underbrace{\varphi_0(q)}_{|q|\leq\frac{1}{2}} + \underbrace{\varphi_1(q)}_{\frac{1}{2}\leq|q|\leq 1} \quad . \tag{17}$$

Integration over $\varphi_1(q)$ ("*thinning out*") leads to the effective action $\overline{S}\,(\varphi_0)$:

$$Z \;=\; \prod_{|q|\leq\frac{1}{2}} \int d\varphi_0(q) \prod_{\frac{1}{2}\leq|q|\leq 1} \int d\varphi_1(q)\, e^{-S(\varphi_0\,+\,\varphi_1)} \quad =$$

$$= \prod_{|q| \le \frac{1}{2}} \int d\varphi_0(q) \; e^{-\overline{S}\,(\varphi_0)} \quad .$$

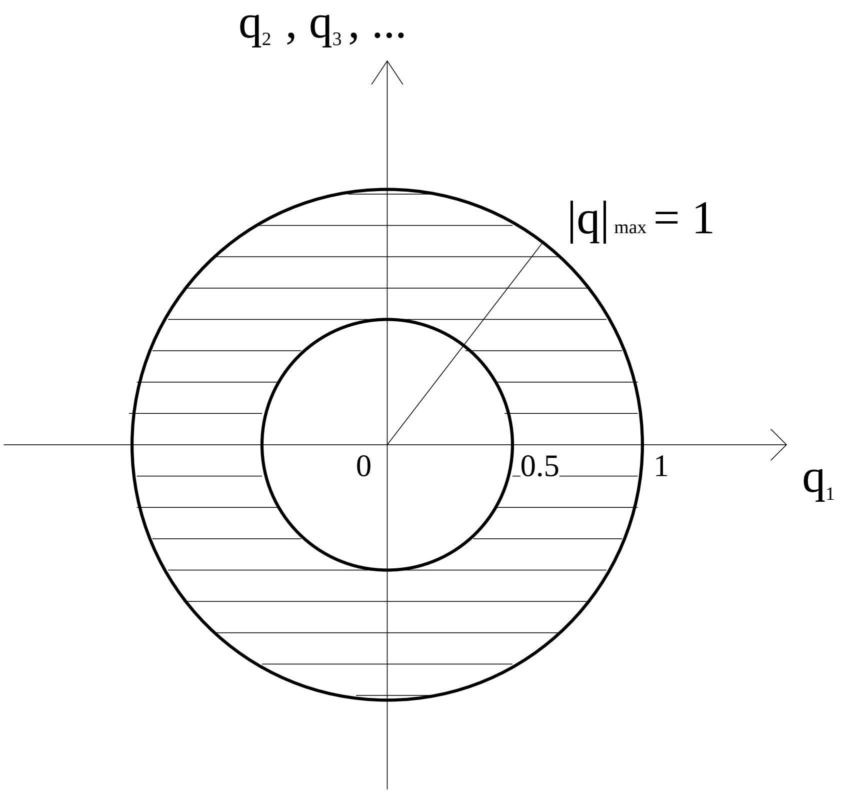

Figure 6: Range of q (The shaded region is integrated out by one RG transformation.).

Now one replaces the dimensionless momentum q by $q' = 2\,q$ ("*relabelling*") with

$$0 \le |q'| \le 1 \quad .$$

This leads to the following change in the field variables ("*rescaling*"):

$$\varphi'(q') = \underbrace{\zeta^{-1}}_{\text{const}} \; \varphi_0(q) \quad . \tag{18}$$

For the generating functional one obtains

$$Z = \prod_{|q'|\le 1} \int d\varphi'(q')\, e^{-S'(\varphi')} =$$

$$\stackrel{!}{=} \prod_{|q|\le 1} \int d\varphi\,(q)\, e^{-S'(\varphi)} \quad . \tag{19}$$

In the second step primed fields and integration variables were just replaced by unprimed symbols. The net effect of the RG transformation between equation (16) and (19) is just the following change in the action:

$$S\,(\varphi) \stackrel{RG}{\longrightarrow} S'(\varphi) \quad .$$

The low energy (long distance) correlations remain unchanged by the RG transformation as one can check ($|q_1|, |q_2| \ll 1$):

$$< \varphi\,(q_1)\, \varphi\,(q_2) >|_S \quad =$$

$$= \frac{1}{Z} \prod_{|q|\le 1} \int d\varphi\,(q)\, \varphi\,(q_1)\, \varphi\,(q_2)\, e^{-S(\varphi)} \quad =$$

$$= \frac{1}{Z} \prod_{|q|\le \frac{1}{2}} \int d\varphi_0(q) \;\; \varphi_0(q_1)\, \varphi_0(q_2) \prod_{\frac{1}{2}\le|q|\le 1} \int d\varphi_1(q)\, e^{-S(\varphi_0+\varphi_1)} =$$

$$= \frac{1}{Z} \prod_{|q|\le \frac{1}{2}} \int d\varphi_0(q) \;\; \varphi_0(q_1)\, \varphi_0(q_2)\, e^{-\overline{S}\,(\varphi_0)} \quad \stackrel{\text{eq. (18)}}{=}$$

$$= \frac{1}{Z}\, \zeta^2 \prod_{|q'|\le 1} \int d\varphi'(q') \;\, \varphi'(\frac{q_1}{2})\, \varphi'(\frac{q_2}{2})\, e^{-S'(\varphi')} \quad .$$

That is:

$$\underbrace{< \varphi\,(q_1)\,\varphi\,(q_2) >|_S}_{\sim\,\delta(q_1+q_2)\;G^{(2)}(q_1)\big|_S} \quad = \; \zeta^2 \underbrace{< \varphi'(2\,q_1)\,\varphi'(2\,q_2) >|_{S'}}_{\sim\,\delta(2\,q_1+2\,q_2)\;G^{(2)'}(2\,q_1)\big|_{S'}} \quad .$$

Using $\delta(2\,q) = 2^{-d}\delta(q)$ one gets a relation for the two point Greens functions:

$$G^{(2)}(q)\Big|_S = \zeta^2\,2^{-d}\,G^{(2)'}(2\,q)\Big|_{S'} \quad . \tag{20}$$

What we now see is the following: If the propagator has a pole at $q^2 = -m^2$ ($m \ll 1$), i.e. $G^{(2)}(q) \sim \left(q^2+m^2\right)^{-1}$, then the pole will be shifted by the RG transformation to the position $m' = 2\,m$, as

$$G^{(2)'}(q) \quad \sim \quad \frac{1}{\left(\frac{q}{2}\right)^2 + m^2} \quad \sim \quad \frac{1}{q^2 + (2\,m)^2} \quad \sim \quad \frac{1}{q^2 + (m')^2} \quad .$$

Although the dimensionful mass $m_{df} = \frac{m}{\Lambda^{cut}}$ and so the dimensionful correlation length $\xi_{df} \sim (m_{df})^{-1}$ is not changed by the RG transformation, we have the following changes for the action $S\,(\varphi)$, the dimensionless mass m, the dimensionless correlation length $\xi \sim m^{-1}$ and the cutoff Λ^{cut} itself:

$$\begin{aligned} S\,(\varphi) &\xrightarrow{RG} S'(\varphi) \\ m &\xrightarrow{RG} 2\,m \\ \xi &\xrightarrow{RG} \frac{\xi}{2} \\ \Lambda^{cut} &\xrightarrow{RG} \frac{\Lambda^{cut}}{2} \quad . \end{aligned} \tag{21}$$

3.2 RG transformation in configuration space (lattice)

On the lattice the generating functional Z looks as follows:

$$Z = \prod_n \int d\varphi_n \, e^{-S(\varphi)} \quad , \tag{22}$$

where n runs over all lattice sites. When performing a *block transformation*, i.e. averaging over groups of the original lattice sites labelled by n, which are now collected into blocks labelled by the block index n_B (see figure 7), one does the following with the generating functional:

$$\begin{aligned} Z &= \prod_n \int d\varphi_n \overbrace{\prod_{n_B} \int d{\varphi'}_{n_B} \, \delta({\varphi'}_{n_B} - b \sum_{n \in n_B} \varphi_n)}^{1} \, e^{-S(\varphi)} &= \\ &= \prod_{n_B} \int d{\varphi'}_{n_B} \prod_n \int d\varphi_n \, \delta({\varphi'}_{n_B} - b \sum_{n \in n_B} \varphi_n) \, e^{-S(\varphi)} &= \\ &= \prod_{n_B} \int d{\varphi'}_{n_B} \, e^{-S'(\varphi')} \quad , \end{aligned}$$

where

$$e^{-S'(\varphi')} = \prod_n \int d\varphi_n \, \delta({\varphi'}_{n_B} - b \sum_{n \in n_B} \varphi_n) \, e^{-S(\varphi)} \quad . \tag{23}$$

Here n_B extends over all blocks, b is a free parameter, which corresponds to the rescaling parameter ζ in eq. (18), and $S'(\varphi')$ is the resulting effective action after performing the block transformation.

Thus the lattice unit $a \sim (\Lambda^{cut})^{-1}$ transforms under the considered RG transformations (or equivalently under the block transformations) in the following way:

$$a \xrightarrow{RG} 2\,a = a' \quad .$$

The dimensionful and dimensionless correlation length transforms as in eq. (21):

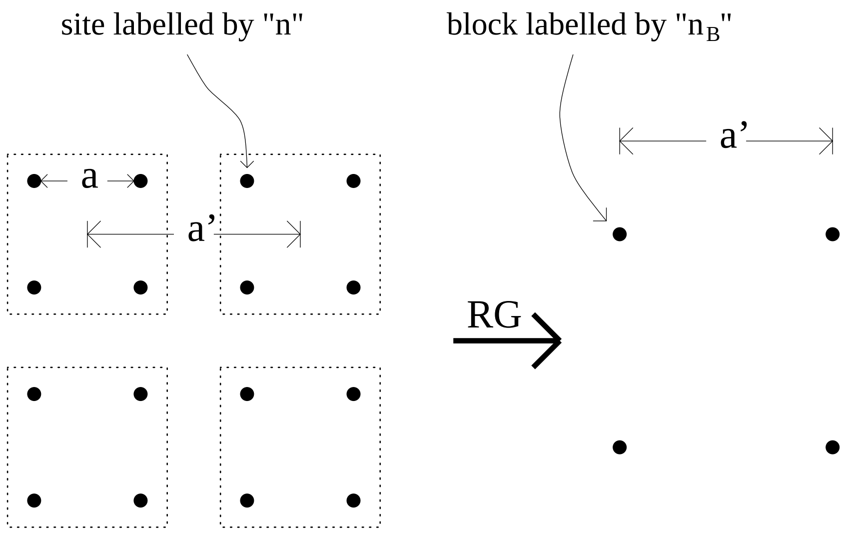

Figure 7: Lattice before and after block transformation.

$$
\begin{aligned}
\xi_{df} &\xrightarrow{RG} \xi_{df} \\
\xi &\xrightarrow{RG} \frac{\xi}{2} \quad .
\end{aligned}
$$

4 Fixed Point (FP), behavior in the vicinity of a FP

Let us see, how the action is changed by RG transformations. The transformed action $S'(\Phi')$, in general, will contain all kinds of different interactions, even if the original action $S(\Phi)$ had a simple form.

To begin with, we take an action of the general form

$$S(\Phi) = \sum_\alpha K_\alpha\, \theta_\alpha(\Phi) \quad ,$$

where $\theta_\alpha(\Phi)$ $(\alpha = 1, 2, \ldots)$ denotes all kinds of different interaction terms like

$$\theta_\alpha(\Phi) : \qquad \int_x \Phi^2(x) \quad , \quad \int_x \Phi^4(x) \quad ,$$

$$\int_x (\partial_\mu \Phi(x))^2 \quad , \quad \int_x (\partial_\mu \Phi(x))^2\, \Phi^2(x) \quad , \ldots \int_x \Phi^{18}(x) \quad , \quad \ldots$$

and K_α the corresponding dimensionless couplings (all coupling constants can be made dimensionless by multiplying with a suitable power of the (dimensionful) cutoff Λ^{cut}). The transformed action $S'(\Phi')$ is expanded similarly:

$$S'(\Phi') = \sum_\alpha K'_\alpha\, \theta_\alpha(\Phi') \quad .$$

Under repeated RG transformations the set of all couplings and the corresponding dimensionless correlation length changes step by step in the following manner:

$$\{K_\alpha\} \ \stackrel{RG}{\longrightarrow}\ \{K'_\alpha\} \ \stackrel{RG}{\longrightarrow}\ \{K''_\alpha\} \ \stackrel{RG}{\longrightarrow}\ \cdots$$

$$\xi \ \stackrel{RG}{\longrightarrow}\ \frac{\xi}{2} \ \stackrel{RG}{\longrightarrow}\ \frac{\xi}{2^2} \ \stackrel{RG}{\longrightarrow}\ \cdots \quad .$$

The RG transformation can have *fixed points* $\{K^*_\alpha\}$ (FP) in the space of couplings which have the property

$$\{K_\alpha^*\} \quad \xrightarrow{RG} \quad \{K_\alpha^*\} \quad .$$

As the dimensionless correlation length has to decrease at each RG step, at a FP it has to be infinity, i.e. $\xi = \infty$ (the other possibility $\xi = 0$ is not interesting for us). A set of points in the space of couplings K_α, where $\xi = \infty$, forms a hyper–surface which is called *critical surface.* Since $\xi \xrightarrow{RG} \frac{1}{2}\xi$, an RG transformation drives the point $\{K_\alpha\}$ *away* from a critical surface, except when $\{K_\alpha\}$ is *on* the critical surface.

Let us consider now the behavior of the action under RG transformation in the vicinity of a FP: Assume there is a FP $\{K_\alpha^*\}$ on the critical surface. Take a point $\{K_\alpha\}$ in the neighborhood of the fixed point, i.e. $\Delta K_\alpha = K_\alpha - K_\alpha^*$ is small. Perform now an RG transformation

$$\{K_\alpha\} \quad \xrightarrow{RG} \quad \{K'_\alpha\} \qquad , \qquad \Delta K'_\alpha = K'_\alpha - K_\alpha^* \quad .$$

We can now express $\Delta K'_\alpha$ in terms of ΔK_α by an expansion

$$\Delta K'_\alpha = \Delta K'_\alpha(\{K\}) = \underbrace{\Delta K'_\alpha(\{K^*\})}_{0} + \sum_\beta \frac{\partial}{\partial K_\beta} \Delta K'_\alpha(\{K^*\}) \cdot \Delta K_\beta + O\left((\Delta K)^2\right)$$

An RG transformation, which takes only into account the linear term of this ΔK–expansion, is usually called *linearized RG transformation.* With the definition:

$$T_{\alpha\beta} = \frac{\partial}{\partial K_\beta} \Delta K'_\alpha(\{K\})\bigg|_{K=K^*}$$

the equation can be rewritten considering only the linear term in the ΔK–expansion

$$\Delta K'_\alpha = \sum_\beta T_{\alpha\beta}\, \Delta K_\beta \quad .$$

Diagonalization of $T_{\alpha\beta}$ gives the eigenvectors h^a $(a = 1, 2, \ldots)$ and the corresponding eigenvalues λ^a [2]. We distinguish three cases:

[2] Actually, this step requires more care: $T_{\alpha\beta}$ is not a symmetric matrix, in general. For the associated technical complications see, for example, ref. [Wil74].

- $|\lambda^a| > 1:$ The eigenvalue is called *relevant*.
- $|\lambda^a| = 1:$ The eigenvalue is called *marginal*.
- $|\lambda^a| < 1:$ The eigenvalue is called *irrelevant*.

The reason for these names is given by the following. Using the notation

$$h^a(\Phi) = \sum_\alpha h^a_\alpha \, \theta_\alpha(\Phi) \quad ,$$

the action can be expanded in the eigenbasis $h^a(\Phi)$:

$$S\,(\Phi) = S^*(\Phi) + \sum_a c^a \, h^a(\Phi) \quad ,$$

where c^a are the corresponding expansion coefficients. Repeated application of RG transformations changes the action close to the FP in the following way:

$$S\,(\Phi) \stackrel{RG}{\longrightarrow} S^*(\Phi) + \sum_a \lambda^a \, c^a \, h^a(\Phi) \stackrel{RG}{\longrightarrow} S^*(\Phi) + \sum_a (\lambda^a)^2 \, c^a \, h^a(\Phi) \stackrel{RG}{\longrightarrow} \ldots \quad .$$

What one sees is that on one hand there is a part $S^*(\Phi)$ in the action, which is unchanged by successive RG transformations. $S^*(\Phi)$ is called the *fixed point action*. On the other hand, the interaction terms associated with an irrelevant eigenvalue, the so called *irrelevant interactions* or *irrelevant operators* are suppressed by each RG transformation step, while terms associated with a relevant eigenvalue (*relevant interactions* or *relevant operators*) are enhanced. For terms belonging to marginal eigenvalues it is not immediately clear what will happen. One has to look to higher order terms in the ΔK–expansion which determine whether the corresponding interaction term will be suppressed or enhanced by RG transformations or whether this term has to be added to the fixed point action.

Let us for instance consider the $d = 3\,(= 2 + 1)$ scalar field theory (see figure 8). In this case two FPs (FP$^{(1)}$ and FP$^{(2)}$) can be identified which have to lie in the critical plane. FP$^{(2)}$ is positioned at the origin of the coupling space. This FP is called *Gaussian FP*. The Gaussian FP itself (here and in models with other field content) describes massless free field theories, but its neighbourhood might reveal more interesting physical properties. As one can see in figure 8,

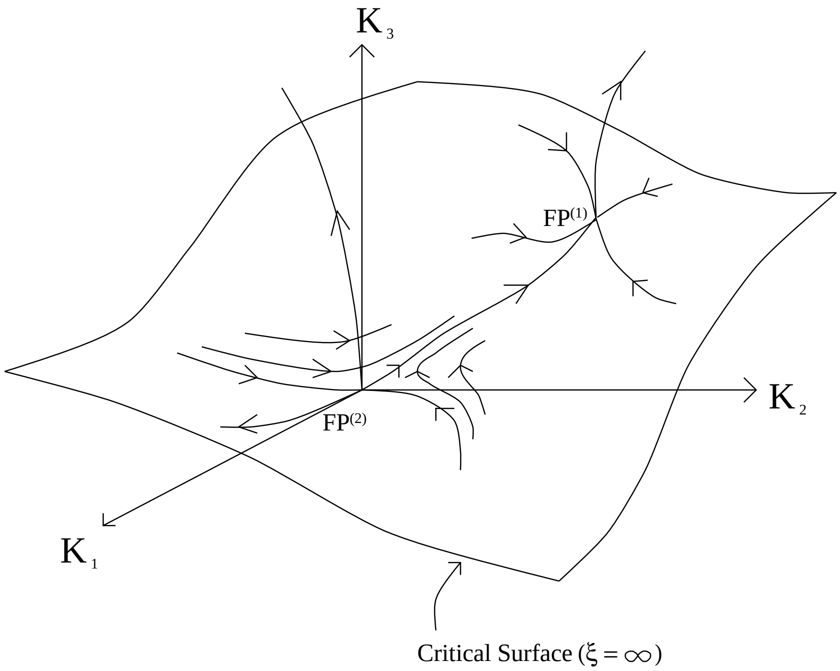

Figure 8: Coupling space and RG trajectories of the $d = 3$ scalar FT.

at each FP there are attractive (irrelevant) and repulsive (relevant) eigendirections with respect to RG transformations. At both fixed points there is a relevant direction running out of the critical surface. In addition, there is one trajectory leading from the Gaussian FP ($FP^{(2)}$) to $FP^{(1)}$. This shows that the dimension of the basin of attraction of $FP^{(2)}$ is one less than the one of $FP^{(1)}$. That means to come to the domain of $FP^{(2)}$ one has to tune one additional parameter.

The number of different fixed points in a d dimensional scalar field theory grows as d is decreased towards 2. On the other hand, in $d = 4$, there is only one FP identified until now.

5 Gaussian FP

5.1 The Gaussian FP in scalar FTs

In the last section it was mentioned that in four Euclidean dimensions ($d = 4 = 3+1$) only one FP can be identified for a scalar FT. This FP is the Gaussian FP which exists in any dimension. The Gaussian FP has an enormous importance, especially in QFT, so we shall investigate its properties in some detail.

To identify the Gaussian FP in scalar FTs, we start with a simple Gaussian scalar theory, i.e., an action which is quadratic in the field

$$S = \frac{1}{2} \int_{k=0}^{\Lambda^{cut}} \tilde{\Phi}\left(k\right) \tilde{\Phi}\left(-k\right) \tilde{\rho}(k) \quad .$$

Here $\tilde{\rho}(k)$ is a general function which can be expanded in increasing powers of k:

$$\tilde{\rho}(k) = \tilde{r}_0 + \underbrace{1}_{\text{convention!}} \cdot \sum_{\mu=1}^{d} k_\mu k_\mu + \tilde{\alpha} \left(\sum_{\mu=1}^{d} k_\mu k_\mu \right)^2 + \tilde{\beta} \sum_{\mu=1}^{d} k_\mu k_\mu k_\mu k_\mu + \ldots \quad .$$

By rescaling the integration variable $\tilde{\Phi}\left(k\right)$ in the path integral one can arrange that the coefficient of the term $\sum_\mu k_\mu k_\mu$ is 1. We shall take this convention. As in section 3 we now introduce dimensionless quantities with the help of the cutoff Λ^{cut}

$$\begin{aligned}
q_\mu &= \frac{k_\mu}{\Lambda^{cut}} \quad , \quad |q| \leq 1 \\
\varphi\left(q\right) &= \tilde{\Phi}\left(k\right) \cdot \left(\Lambda^{cut}\right)^{\frac{d+2}{2}} \\
\rho\left(q\right) &= \tilde{\rho}\left(k\right) \cdot \left(\Lambda^{cut}\right)^{-2} \\
r_0 &= \tilde{r}_0 \cdot \left(\Lambda^{cut}\right)^{-2} \\
\alpha &= \tilde{\alpha} \cdot \left(\Lambda^{cut}\right)^{2} \\
\beta &= \tilde{\beta} \cdot \left(\Lambda^{cut}\right)^{2} \\
&\ldots
\end{aligned}$$

From that we have now

$$
\begin{aligned}
S &= \frac{1}{2}\int_{q=0}^{1} \varphi(q)\,\varphi(-q)\,\rho(q) \\
\rho(q) &= r_0 + \sum_\mu q_\mu q_\mu + \alpha\left(\sum_\mu q_\mu q_\mu\right)^2 + \beta\sum_\mu q_\mu q_\mu q_\mu q_\mu + \dots \quad .
\end{aligned}
$$

Now we perform an RG transformation as in section 3.1. The generating functional is

$$
Z = \int D\varphi(q)\; e^{-S(\varphi)} = \prod_{|q|\le 1} \int d\varphi(q)\;\exp\left\{-\frac{1}{2}\int_{q=0}^{1}\varphi(q)\,\varphi(-q)\,\rho(q)\right\} \quad .
$$

As in section 3.1 we split the fields and the action in low and high momentum parts

$$
\begin{aligned}
\varphi(q) &= \underbrace{\varphi_0(q)}_{0\le|q|\le\frac{1}{2}} + \underbrace{\varphi_1(q)}_{\frac{1}{2}\le|q|\le 1} \\
S(\varphi_0+\varphi_1) &= S(\varphi_0) + S(\varphi_1) \quad ,
\end{aligned}
$$

where the last equation is valid for quadratic actions only. Integration over φ_1 gives for the generating function (up to a multiplicative constant)

$$
\begin{aligned}
Z &= \text{const}\cdot\prod_{|q|\le\frac{1}{2}}\int d\varphi_0(q)\;\exp\left\{-\frac{1}{2}\int_{q=0}^{\frac{1}{2}}\varphi_0(q)\,\varphi_0(-q)\,\rho(q)\right\} = \\
&= \text{const}\cdot\prod_{|q'|\le 1}\int d\varphi'(q')\;\exp\left\{-\frac{1}{2}\zeta^2\,2^{-d}\int_{q'=0}^{1}\varphi'(q')\,\varphi'(-q')\,\rho(\frac{q'}{2})\right\} = \\
&= \text{const}\cdot\prod_{|q'|\le 1}\int d\varphi'(q')\; e^{-S'(\varphi')} \quad .
\end{aligned}
$$

As usual we rescaled the dimensionless momenta and fields:

$$\varphi'(q') = \zeta^{-1} \varphi_0(q) \quad , \quad q' = 2\,q \quad , \quad 0 \le |q'| \le 1 \quad .$$

So we have

$$S\,(\varphi) \quad \xrightarrow{RG} \quad S'(\varphi') = \frac{1}{2}\,\zeta^2\, 2^{-d} \int_{q'=0}^{1} \varphi'(q')\,\varphi'(-q')\,\rho\,(\frac{q'}{2})$$

$$\rho\,(q) \quad \xrightarrow{RG} \quad \rho\,\left(\frac{q}{2}\right)\,\zeta^2\, 2^{-d} =$$

$$= \zeta^2\, 2^{-d} \cdot \left\{ r_0 + \frac{1}{4} \sum_\mu q_\mu q_\mu + \frac{1}{16}\,\alpha \left(\sum_\mu q_\mu q_\mu\right)^2 + \frac{1}{16}\,\beta \sum_\mu q_\mu q_\mu q_\mu q_\mu + \ldots \right\} \quad .$$

To satisfy the convention that the coefficient of the $\sum_\mu q_\mu q_\mu$–term in the action should be 1 we choose:

$$\zeta = 2^{\frac{d+2}{2}} \quad .$$

From that we get

$$\begin{array}{lllllll}
r_0 & \xrightarrow{RG} & r_0' = 4\,r_0 & \xrightarrow{RG} & r_0'' = 4^2\,r_0 & \xrightarrow{RG} & \ldots \\
\alpha & \xrightarrow{RG} & \alpha' = \frac{1}{4}\,\alpha & \xrightarrow{RG} & \alpha'' = \left(\frac{1}{4}\right)^2 \alpha & \xrightarrow{RG} & \ldots \\
\beta & \xrightarrow{RG} & \beta' = \frac{1}{4}\,\beta & \xrightarrow{RG} & \beta'' = \left(\frac{1}{4}\right)^2 \beta & \xrightarrow{RG} & \ldots \\
\ldots & & & & & &
\end{array}$$

We see that the only relevant operator in the action is that with the coefficient r_0 (eigenvalue: $\lambda = 4$), all other interactions are irrelevant, i.e.:

r_0	relevant	$\lambda = 4$
α	irrelevant	$\lambda = \frac{1}{4}$
β	irrelevant	$\lambda = \frac{1}{4}$
all other	irrelevant	$\lvert\lambda\rvert < 1$

Repeated RG transformations therefore lead to an FP action which is that of a *free* scalar FT of massless ($\xi = \infty$) particles:

$$S^*(\varphi) \;=\; \frac{1}{2}\int_{q=0}^{1} \varphi\,(q)\,\varphi\,(-q)\,\rho^*(q)$$

$$\rho^*(q) \;=\; q^2 \quad .$$

5.2 Classification of operators in the vicinity of the Gaussian FP

In the previous section we considered couplings which are quadratic in the fields. Let us generalize this by considering an extended coupling constant space in the vicinity of the Gaussian FP. Add for example to the Gaussian FP action some terms like:

$$\tilde{\lambda}_4 \int_x \Phi^4(x) \quad , \quad \tilde{\lambda}_6 \int_x \Phi^6(x) \quad , \quad \ldots \quad .$$

This leads in dimensionless momentum space to following terms in the action with dimensionless couplings r_0, λ_4, λ_6 ($\ll 1$):

$$S \;=\; \int_q \frac{1}{2}\left(q^2 + r_0\right)\varphi\,(q)\,\varphi\,(-q) +$$

$$+ \quad \lambda_4 \int_{q_1 \dots q_4} \varphi(q_1)\, \varphi(q_2)\, \varphi(q_3)\, \varphi(q_4)\ \delta(q_1 + q_2 + q_3 + q_4) +$$

$$+ \quad \lambda_6 \int_{q_1 \dots q_6} \varphi(q_1)\ \dots\ \varphi(q_6)\ \delta(q_1 + \dots + q_6) + \dots \quad .$$

Straightforward calculation gives the following transformation behavior of the coupling *close* to the Gaussian FP:

$$r_0 \ \xrightarrow{RG}\ (r_0 + 6\,\lambda_4\, c + 15\,\lambda_6\, c^2)\, \zeta^2\, 2^{-d}$$

$$\lambda_4 \ \xrightarrow{RG}\ (\lambda_4 + 15\,\lambda_6\, c)\, \zeta^4\, 2^{-3d}$$

$$\lambda_6 \ \xrightarrow{RG}\ \lambda_6\, \zeta^6\, 2^{-5d}$$

with

$$c = \int_{k=\frac{1}{2}}^{1} \frac{1}{k^2 + r_0} \quad .$$

The eigenvalues λ^a and the eigenvectors h^a of this linear transformation are therefore

$$\begin{array}{lll} \lambda^{(1)} = 2^2 & h^{(1)} = \Phi^2 & \dim \Phi^2(x) = d - 2 \\ \lambda^{(2)} = 2^{4-d} & h^{(2)} = \Phi^4 + a\,\Phi^2 & \dim \Phi^4(x) = 2d - 4 \\ \lambda^{(3)} = 2^{6-2d} & h^{(3)} = \Phi^6 + b_1\,\Phi^4 + b_2\,\Phi^2 & \dim \Phi^6(x) = 3d - 6 \end{array}$$

In the last column we gave the engineering dimension of the highest power in the eigenvector. The general result is the following:

To every simple operator $\theta(x)$ with engineering dimension d_θ there corresponds an eigenoperator, whose highest dimensional element is $\theta(x)$, and the corresponding eigenvalue is:

$$\lambda^\theta = 2^{d \,-\, d_\theta} \quad .$$

Some examples:

$\theta(x)$	d_θ	λ^θ
$\Phi^2(x)$	$d-2$	4
$\Phi^4(x)$	$2d-4$	2^{4-d}
$\Phi^6(x)$	$3d-6$	2^{6-2d}
$\left(\partial^2\,\Phi(x)\right)^2$	$d+2$	$\frac{1}{4}$

In 3 and 4 Euclidean dimensions the upper result leads to the following statements about the relevance of special operators in linear order in the ΔK–expansion:

$d = 3$	$O(\Delta K)$
$\Phi^2\,,\ \Phi^4$	relevant
Φ^6	marginal
all other operators	irrelevant

$d = 4$	$O(\Delta K)$
Φ^2	relevant
Φ^4	marginal
all other operators	irrelevant

The fate of the marginal operators Φ^6 $(d=3)$ and Φ^4 $(d=4)$ is decided by the $O\left((\Delta K)^2\right)$ terms. In both cases they become *weakly irrelevant.*

5.3 A short look at the Yang Mills interaction

If the form of the RG transformation respect the symmetries of the original action, then the action after the transformation will also have these symmetries.

It might happen that a symmetry of the theory is broken by the regularization. If the corresponding symmetry breaking interactions have *no* projection to the relevant directions of the RG transformation, then the symmetry is automatically regained in the continuum limit. The term proportional to β in ρ in the previous section gives an example for this possibility: this rotation symmetry breaking perturbation dies out in the infrared (continuum) limit.

In contrary, if a gauge symmetry is broken, the corresponding non–gauge invariant interactions will be relevant and the gauge symmetry will not come back automatically in the continuum limit. Really, on the Gaussian fixed point there exist several non–gauge invariant relevant (marginal) operators in a $d = 4$ Yang–Mills theory ($\lambda =$ eigenvalue of RG transformation):

	$O(\Delta K)$	
$A^a_\mu A^a_\mu$	relevant	$(\lambda = 4)$
$A^a_\mu A^a_\mu A^b_\nu A^b_\nu$	marginal	$(\lambda = 1)$
$\partial_\mu A_\mu A_\nu A_\nu$	marginal	$(\lambda = 1)$

If one restricts oneself to gauge invariant operators, it turns out that the lowest dimensional gauge invariant operator:

$$\frac{1}{4\,g^2} F^a_{\mu\nu} F^a_{\mu\nu} \qquad \text{with} \qquad d_\theta = 4$$

is *marginal* to first order and becomes *weakly relevant* to second order in the ΔK–expansion while all other gauge invariant operators are *irrelevant*. The coupling g is the only interaction which survives the infrared limit. This coupling is growing as the momentum scale is decreased and the other way around ("asymptotic freedom").

The simple topology of the coupling space for $d = 4$ Yang–Mills theory is displayed in figure 9 where $g, c_1, c_2, \ldots$ denote the relevant and irrelevant couplings respectively.

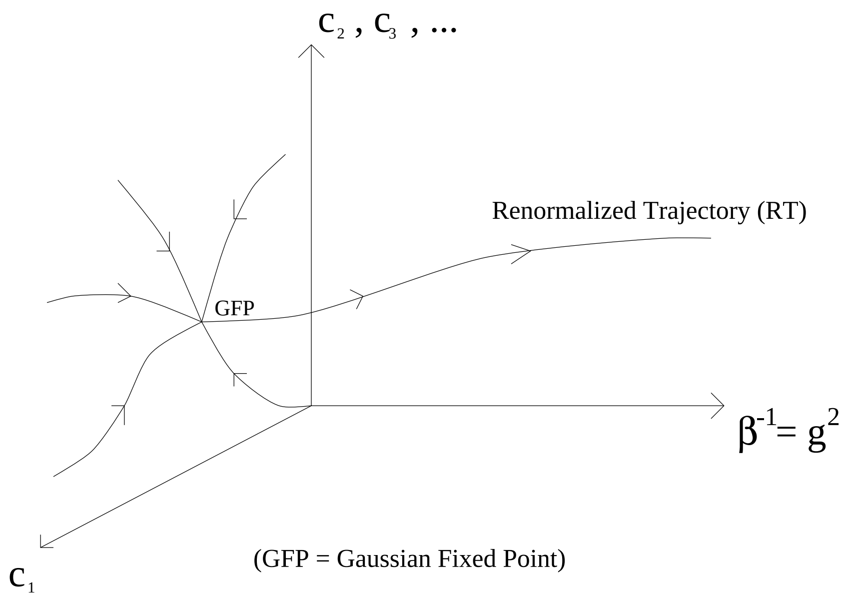

Figure 9: Gaussian Fixed Point in $d = 4$ Yang–Mills theory.

6 The perfect lattice regularized free scalar theory; perfect Laplace operator

Take, for concreteness, the $d = 2$ Euclidean space continuum action

$$S = \int d^2x \, \frac{1}{2} \sum_{\mu=1}^{2} \partial_\mu \Phi(x) \, \partial_\mu \Phi(x), \quad . \tag{24}$$

The standard way to put this theory on the lattice is to replace the derivative by a discrete difference. Suppressing the lattice unit dependence ('$a = 1$') one obtains:

$$\begin{aligned} -\int d^2x \, \frac{1}{2} \sum_{\mu=1}^{2} \Phi(x) \Delta^{cont} \Phi(x) \quad &= \\ \overset{i.b.p.}{=} \int d^2x \, \frac{1}{2} \sum_{\mu=1}^{2} \partial_\mu \Phi(x) \, \partial_\mu \Phi(x) \quad &\to \quad \sum_{n,\hat{\mu}} \frac{1}{2} \left(\Phi_{n+\hat{\mu}} - \Phi_n \right)^2 \, , \end{aligned} \tag{25}$$

where $\hat{\mu}$ is the unit vector in 1 or 2 direction. This corresponds to the Laplace operator on the lattice

$$\left(\Delta^{lattice}_{standard} \Phi \right)_n = 4\Phi_n - \sum_{\hat{\mu}=1}^{2} \left(\Phi_{n+\hat{\mu}} + \Phi_{n-\hat{\mu}} \right) \, . \tag{26}$$

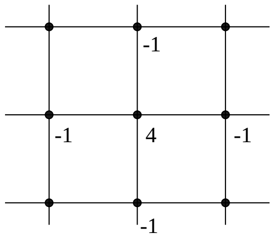

Figure 10: Standard Laplace operator on the lattice.

This is the familiar $4, -1, -1, -1, -1$ rule for the lattice Laplace operator. It is clear, however, that $\Delta^{lattice}_{standard}$ is only an approximation to the continuum $\Delta^{cont} = \sum_\mu \partial_\mu \partial_\mu$, and by solving $\Delta^{lattice}_{standard}\Phi = 0$ the predictions will be distorted. For small a, the distortion is $\mathrm{O}(a^2)$[3].

Similarly, the corresponding field theory gives predictions which are contaminated by lattice artifacts. Let us determine the spectrum. In the continuum we have

$$\begin{aligned}
\int dx_1\, e^{-ipx_1} D^{cont}(x_1, x_2) &= \int dx_1\, e^{-ipx_1} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{e^{ik_1x_1} e^{ik_2x_2}}{k_1^2 + k_2^2} \\
&= \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \frac{e^{ik_2x_2}}{p^2 + k_2^2} \\
&= \frac{1}{2p} e^{-|p|x_2}, \qquad (27)
\end{aligned}$$

where D^{cont} is the Greens function in the continuum (which obeys $\Delta^{cont} D^{cont}(x_1, x_2) = \delta(x_1) \cdot \delta(x_2)$) and $\int dx_1\, e^{-ipx_1}$ projects into the momentum ($= p$) channel. Comparing this with the expected behavior of the propagator

$$\int dx_1\, e^{-ipx_1} D^{cont}(x_1, x_2) \overset{x_2 \to \infty}{\sim} e^{-x_2 E(p)}, \qquad (28)$$

one observes the expected dispersion relation $E = |p|$.

Things will change, however, if we consider the propagator for the standard lattice Laplace operator (see e.g. [Cre83]). By calculating the lattice action in Fourier space

$$\begin{aligned}
\sum_{n,\hat{\mu}} \frac{1}{2} \left(\Phi_{n+\hat{\mu}} - \Phi_n\right)^2 &= \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{1}{2} \sum_\mu \left(e^{ik_\mu} - 1\right)\left(e^{-ik_\mu} - 1\right) \tilde{\Phi}(k)\tilde{\Phi}(-k) \\
&= 2 \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \sum_\mu sin^2\left(\frac{k_\mu}{2}\right) \tilde{\Phi}(k)\tilde{\Phi}(-k), \qquad (29)
\end{aligned}$$

[3]Boundary effects can also produce $\mathrm{O}(a)$ corrections.

one can read off the propagator for $\Delta^{lattice}_{standard}$. Restoring the a-dependence, one gets for the propagator for small k ($k << \frac{1}{a}$):

$$\frac{1}{\frac{4}{a^2}\sum_\mu sin^2\left(\frac{k_\mu a}{2}\right)} = \frac{1}{k^2 + O(a^2)}. \tag{30}$$

This corresponds to the dispersion relation

$$E(p) = |p| + O(p^3) \tag{31}$$

for the standard lattice action for small $|p|$. The $O((p^3)$-terms represents lattice artifacts. As one can see in figure (11) the standard lattice dispersion relation

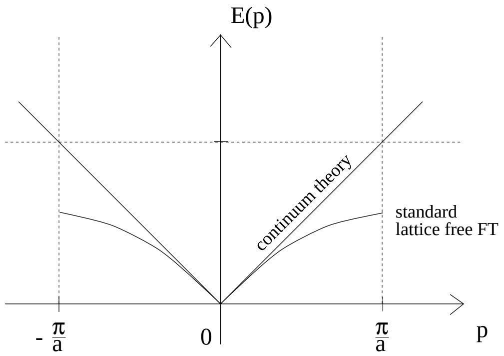

Figure 11: Dispersion relations for continuum theory and standard lattice theory.

approximates the continuum relation very well for small $|p|$ but gets very bad near the edge of the Brillouin zone.

It might sound surprising, but there exists a lattice version of the free scalar theory which is free of lattice artifacts and most notably, it gives back the exact spectrum of the continuum theory. This formulation defines a perfect Laplace operator also. Even more, this perfect lattice representation of the model is sufficiently simple to be interesting in practical calculations.

This perfect action is the FP action of the model obtained on the lattice. To find this action, let us start with a general quadratic form in $d = 2$ dimensions:

$$S = \frac{1}{2}\sum_{n,r} \rho(r)\, \phi_n \phi_{n+r}, \tag{32}$$

where $\rho(r)$ are the coupling constants and $n = (n_1, n_2)$, $r = (r_1, r_2)$. Now look at the RG transformation

$$\prod_n \int d\phi_n \, e^{-\frac{1}{2}\sum_{n,r} \rho(r)\, \phi_n \phi_{n+r}} \prod_{n_B} \delta\left(\chi_{n_B} - \frac{1}{4}\sum_{n \in n_B} \phi_n\right). \tag{33}$$

We generalize this transformation by the replacement

$$\delta\left(\chi_{n_B} - \frac{1}{4}\sum_{n\in n_B}\phi_n\right) \longrightarrow \exp\left\{-2\kappa\left(\chi_{n_B} - \frac{1}{4}\sum_{n\in n_B}\phi_n\right)^2\right\}, \tag{34}$$

where κ is a free parameter. For $\kappa \to \infty$ we get the δ-function blocking, for finite κ it is a (legitime) generalization. We have

$$\begin{aligned} & e^{-\frac{1}{2}\sum_{n_B,r_B}\rho'(r)\,\chi_{n_B}\chi_{n_B+r_B}} = \\ & = \prod_n \int d\phi_n \, e^{-\frac{1}{2}\sum_{n,r}\rho(r)\,\phi_n\phi_{n+r} - 2\kappa\left(\chi_{n_B} - \frac{1}{4}\sum_{n\in n_B}\phi_n\right)^2} . \end{aligned} \tag{35}$$

The problem is to find a a specific set of couplings $\rho^*(r)$ such that if $\rho(r) = \rho^*(r)$, then $\rho'(r) = \rho^*(r)$ also (fixed point). Since we have to perform Gaussian integrals only, this problem can be solved relatively easily. We will only give the result in Fourier space:

$$\frac{1}{\tilde{\rho}^*(q)} = \sum_l \frac{1}{(q+2\pi l)^2}\prod_{i=0}^{1}\frac{\sin^2\frac{q_i}{2}}{\left(\frac{q_i}{2}+\pi l_i\right)^2} + \frac{1}{3\kappa},$$

where $l = (l_0, l_1)$ is an integer vector and $\sum_l = \sum_{l_0=-\infty}^{\infty}\sum_{l_1=-\infty}^{\infty}$

$$\text{and } \rho^*(r) = \int_{-\pi}^{\pi}\frac{d^2q}{(2\pi)^2}e^{iqr}\tilde{\rho}^*(q), \quad q = (q_1, q_2). \tag{36}$$

One can calculate the spectrum for this perfect Laplace operator and the astonishing thing happens that one has a tower of poles when doing the integration in the k_2-plane. The dispersion relation reads

$$E = |p + 2\pi l_1|, \quad \text{where} \quad l_1 = 0, \pm1, \pm2, \ldots . \tag{37}$$

Therefore we obtain the full continuum spectrum as shown in figure 12.

The properties (most notably the range of interaction) of $\rho(r)$ depend on the free parameter κ. The choice $\kappa \approx 2.$ is optimal. For $\kappa = 2$, $\rho(r)$ decays very rapidly with growing $|r|$, like $\sim \exp(-3.44|r|)$. For this choice $\rho(r)$ is strongly dominated by the nearest neighbor and diagonal couplings while the couplings at distance > 1 are already small. As Table 1 (taken from [Has94]) shows, $\rho(3,3)$ is, for example, 5 orders of magnitude smaller than the nearest neighbor coupling. The choice $\kappa = \infty$ (corresponding to a block transformation with δ–function) gives a considerably larger interaction range, $\rho(r) \sim \exp(-1.45|r|)$.

r	$\rho(r)$	r	$\rho(r)$
(1,0)	-0.61802	(4,0)	$-2.632 \cdot 10^{-6}$
(1,1)	-0.19033	(4,1)	$7.064 \cdot 10^{-7}$
(2,0)	$-1.998 \cdot 10^{-3}$	(4,2)	$1.327 \cdot 10^{-6}$
(2,1)	$-6.793 \cdot 10^{-4}$	(4,3)	$-7.953 \cdot 10^{-7}$
(2,2)	$1.625 \cdot 10^{-3}$	(4,4)	$6.895 \cdot 10^{-8}$
(3,0)	$-1.173 \cdot 10^{-4}$	(5,0)	$-8.831 \cdot 10^{-8}$
(3,1)	$1.942 \cdot 10^{-5}$	(5,1)	$3.457 \cdot 10^{-8}$
(3,2)	$5.232 \cdot 10^{-5}$	(5,2)	$3.491 \cdot 10^{-8}$
(3,3)	$-1.226 \cdot 10^{-5}$	(5,3)	$-3.349 \cdot 10^{-8}$
		(5,4)	$8.408 \cdot 10^{-9}$
		(5,5)	$-1.657 \cdot 10^{-10}$

Table 1: The couplings of the quadratic two–spin interaction terms at a distance $r = (r_0, r_1)$ for the optimal choice of the block transformation with $\kappa = 2$. Note that in our convention, for the standard action the only non–vanishing entry in this list would be $\rho_{ST}(1,0) = -1$.

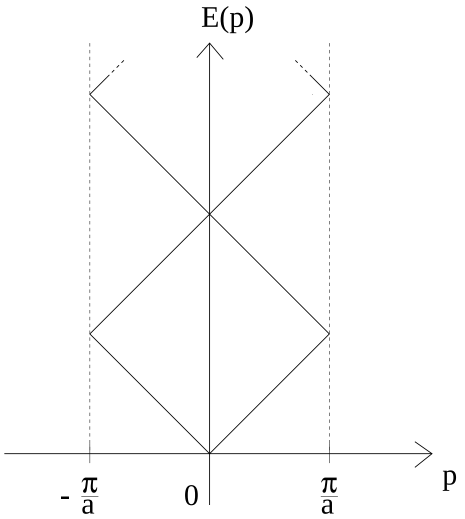

Figure 12: Dispersion relation for the perfect Laplace operator on the lattice.

7 Perfect lattice action in asymptotically free theories

In this section we illustrate the idea of improved lattice actions. As Yang-Mills theory or QCD in 4 dimensions are technically quite complicated, we choose as a toy model the non-linear σ-model in $d = 2$ dimensions. More details can be found in [Has94]. In this section we represent essential parts of this work done by Hasenfratz and Niedermayer.

7.1 Introduction to the non-linear σ-model

The continuum action of the O(3)-symmetric non-linear σ-model in Euclidean space is given by

$$\begin{aligned} S^{cont} &= \frac{\beta}{2}\int d^2x \sum_{\mu=1}^{2} \partial_\mu \vec{\Phi}(\vec{x})\ \partial_\mu \vec{\Phi}(\vec{x}), \\ &\qquad \text{with the constraint : } \quad \vec{\Phi}^2(\vec{x}) = 1, \end{aligned} \tag{38}$$

where $\vec{\Phi}$ is a 3-component vector and $\beta = \frac{1}{g}$, the coupling constant g being dimensionless.

This model has some features common to Yang-Mills theories in 4 dimensions:

- The theory is asymptotically free in g.
- The dynamics of the system generates a mass gap m. We obtain a triplet of particles with mass m.
- Classical instanton solutions exist with scale invariant actions.
- This model has a long history, as this interesting, but relative simple theory always attracted people.

Let us put the theory on the lattice. One obtains the standard lattice formulation of the action (38) by replacing the derivatives with nearest neighbor differences,

$$S^{lattice}_{standard} = \frac{\beta}{2}\sum_{n,\hat{\mu}} \left(\vec{\Phi}_{n+\hat{\mu}} - \vec{\Phi}_n\right)^2 = -\beta \sum_{n,\hat{\mu}} \vec{\Phi}_n \vec{\Phi}_{n+\hat{\mu}} + const, \tag{39}$$

where $n = (n_1, n_2)$ labels the lattice sites and $\hat{\mu}$ is the unit vector in 1 or 2-direction.

7.2 The renormalized coupling g_r

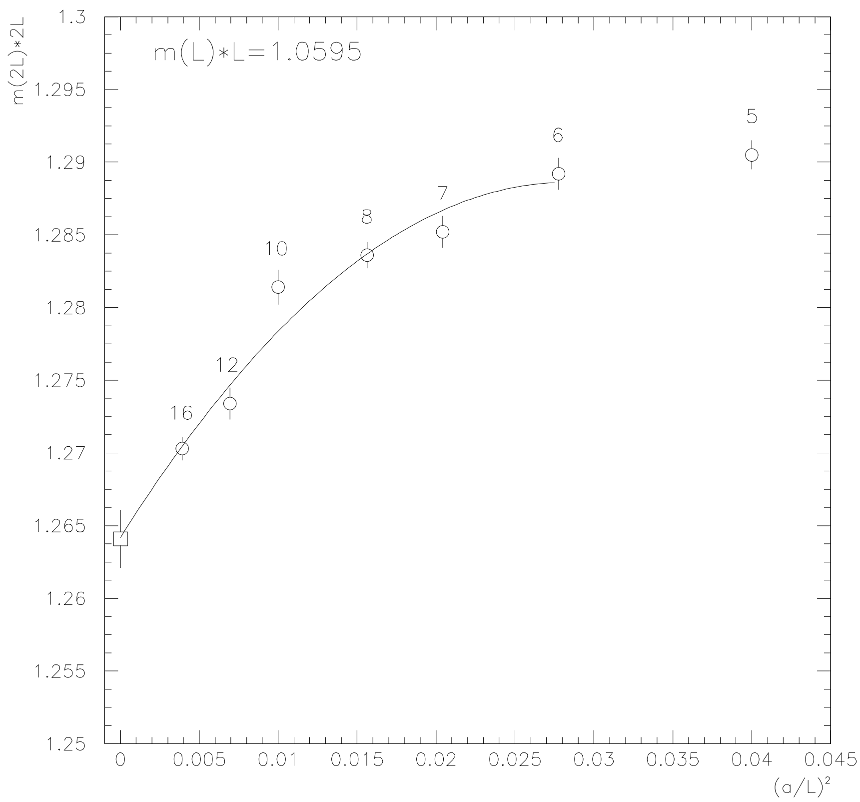

Figure 13: Running coupling in non-linear σ-model: extrapolation to the continuum limit.

An interesting problem is to calculate the renormalized coupling $g_r(L)$ for this model, where L is some physical length scale. For small L ($L << \frac{1}{m}$) this can be done in perturbation theory, because of the asymptotic freedom. The challenging task is, however, to calculate $g_r(L)$ for large L, away from the perturbative regime.

Before we will deal with this problem, we first have to define the renormalized coupling $g_r(L)$. We claim that

$$g_r(L) := m(L) \cdot L \tag{40}$$

is a reasonable definition, where $m(L)$ is the mass gap in a finite, periodic box of length L in space direction (time is kept infinite). We will provide an intuitive argument for the validity of relation (40):

Consider L to be small, then only the constant modes of $\vec{\Phi}$ with respect to x are important. Thus

$$S = \frac{\beta}{2} \int_0^L dx \int dt \sum_{\mu=1}^{2} \left(\partial_\mu \vec{\Phi}(\vec{x}) \right)^2 \quad \stackrel{L \to 0}{\longrightarrow} \quad \frac{L}{2g} \int dt \left(\dot{\vec{\Phi}} \right)^2 . \tag{41}$$

But (41) is just a simple problem of quantum mechanics. It describes a quantum mechanical rotator in 3 dimensions with a moment of inertia $\Theta = \frac{L}{g}$. The energy spectrum is given by

$$E = \frac{l(l+1)}{2\Theta} = \frac{l(l+1)}{2\frac{L}{g}}, \qquad l = 0, 1, 2, \ldots \tag{42}$$

and we obtain the mass gap as the energy of the lowest excitation,

$$m(L) = \frac{g}{L} = \frac{g_r}{L} + O(g^2). \tag{43}$$

After this detour we return to the task of calculating the renormalized coupling in the non-perturbative regime. We consider the work of Lüscher, Weisz and Wolff ([Lüs91]). They start with a value of $g_r(L) = 1.0505$ and want to calculate $g_r(2L)$. Their procedure is as follows. A lattice is introduced with lattice unit a, and for the given $g_r(L) = 1.0505 \quad g_r(2L)$ is calculated on lattices with $\frac{L}{a} = 5, 6, \ldots, 16$ and the results are extrapolated to $\frac{L}{a} = \infty$. This is a non-trivial problem even in $d = 2$. The results of this procedure are shown in figure 13.

The question we want to investigate is, whether we can find an improved lattice action with less severe lattice artifacts.

7.3 RG group flow and the perfect action for the non-linear σ-model

A radical solution would be to use a *perfect* lattice action which is completely free of lattice artifacts. That such perfect actions exist, follows from Wilson's renormalization group (RG) theory ([Wil74], [Wil75] and [Wil80]).

Beyond the basic requirements of O(3) symmetry, locality, correct classical limit, translation and 90°–rotation symmetry, the form of the lattice action is largely arbitrary. It might contain nearest neighbor, next–to–nearest neighbor, etc., even different multi-spin interactions. Let us denote the corresponding

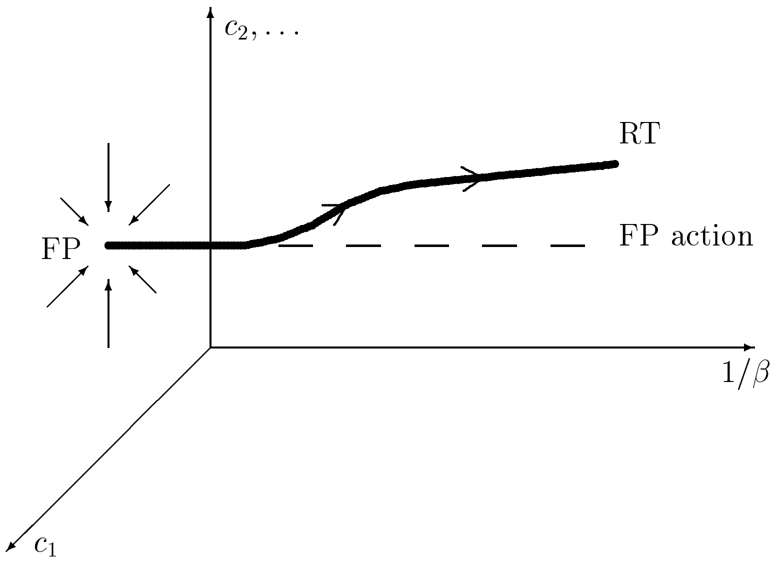

Figure 14: Flow of the couplings under RG transformation in the O(N) non–linear σ–model.

couplings by c_1, $c_2, \ldots$. The action $\beta S(\vec{\Phi})$ is represented by a point in the infinite dimensional space of couplings $(\beta, c_1, c_2, \ldots)$. We shall consider RG transformations in configuration space, namely block transformations with a scale factor of 2. Under repeated block transformations the action moves in this coupling constant space. The expected flow diagram is sketched in figure 14 [Wil80, Has88]. In the $\beta = \infty$ hyperplane (correlation length $\xi = \infty$) there is a fixed point (FP) $c_1^*, c_2^*, \ldots$, whose exact position depends on the details of the block transformation. We shall use the notation $S(\vec{\Phi}; c_1^*, c_2^*, \ldots) = S^*(\vec{\Phi})$ and call $\beta S^*(\vec{\Phi})$ the FP–action. The FP has one marginal and infinitely many irrelevant directions. The marginal operator is S^* itself [Wil80]. Actually, S^* is not exactly marginal, it is weakly relevant. The trajectory which leaves the FP along the weakly relevant direction is called the renormalized trajectory (RT). For large β the RT runs along the FP–action, but for smaller β they do not coincide anymore.

It is easy to see that the points of the RT define perfect actions. The argument goes as follows: At any given β, the point of the RT is connected to the infinitesimal neighborhood of the FP by (infinitely many steps of) exact RG transformations. Since each step increases the lattice unit by a factor of 2, *any* distance at the given β (even 1 lattice unit) corresponds to a long distance close to the FP. The infinitesimal neighborhood of the FP is in the continuum limit, there are no cut–off effects there at long distances. On the other hand, for all the questions which can be formulated in terms of the degrees of freedom after the transformation we get the same answer as before it. So, there are no lattice artifacts at the given β on the RT at any distances.

Since an RG step is a non–critical problem, the FP–action and the actions on the RT in general are expected to be local ([Wil74], [Wil75] and [Wil80]). Locality, however, allows non–negligible interactions over several lattice units in the action. For practical reasons we need more than locality. For practical applicability we have to answer positively the following questions: i. Is it possible to determine $S^*(\vec{\Phi})$ to a good precision? ii. Is $S^*(\vec{\Phi})$ of sufficiently short range? Is the structure of S^* simple enough allowing a parameterization where the number of couplings remains relatively low, O(10–100)? iii. Questions i. and ii. for the points of the RT.

Our pilot study shows that for the non–linear σ–model, using a properly chosen RG transformation, the answer to these questions is 'yes', even if we go down to small correlation lengths . The determination of $S^*(\vec{\Phi})$ is a saddle point problem which requires minimalization over classical fields. The range of interaction in $S^*(\vec{\Phi})$ depends on the RG transformation. By a proper choice of the block–transformation $S^*(\vec{\Phi})$ becomes surprisingly short ranged. $S^*(\vec{\Phi})$ contains multi-spin couplings also but its structure is relatively simple. With O(20) couplings an excellent parameterization can be obtained which works even on coarse configurations, i.e. at small correlation lengths. The problems related to $S^*(\vec{\Phi})$ can be solved partly analytically, which is a special bonus in asymptotically free theories.

7.4 Determining the perfect classical action for the non-linear σ-model

After this long remarks let us look in detail at the RG transformation in configuration space. As we have seen in section 3.2 (equation 23), one naturally introduces block spins as averages over the original degrees of freedom. In our case the effective action would read

$$e^{-\beta' S'(\vec{\Phi}')} \quad = \quad \int_{\vec{\Phi}} \delta \left(\vec{\Phi}'_{n_B} - \frac{\sum_{n\in n_B} \vec{\Phi}_n}{\left| \sum_{n\in n_B} \vec{\Phi}_n \right|} \right) \, e^{-\beta S(\vec{\Phi})},$$

$$\text{where :} \qquad \int_{\vec{\Phi}} = \prod_n \int d\vec{\Phi}_n \delta\left(\vec{\Phi}_n^2 - 1\right). \tag{44}$$

We would like to introduce a parameter in the transformation (44), which can be tuned in order to make the fixed point action short ranged. For this purpose we write the the delta function in (44) as

$$\begin{aligned}\delta\left(\vec{\Phi}'_{n_B}-\frac{\sum_{n\in n_B}\vec{\Phi}_n}{\left|\sum_{n\in n_B}\vec{\Phi}_n\right|}\right)=\\ = const\cdot\lim_{\kappa\to\infty}\exp\left\{\beta\kappa\left[\vec{\Phi}'_{n_B}\sum_{n\in n_B}\vec{\Phi}_n-Y_{Nn}\left(\beta\kappa\cdot\left|\sum_{n\in n_B}\vec{\Phi}_n\right|\right)\right]\right\}.\end{aligned}\tag{45}$$

The function Y_N (for O(N)-symmetry) obeys the relation

$$\int d\vec{\Phi}\,\delta\left(\vec{\Phi}^2-1\right)e^{\vec{\Phi}\vec{b}}=const\cdot Y_N\left(\left|\vec{b}\right|\right)\tag{46}$$

and ensures that the partition function is only changed up to an irrelevant constant by the corresponding RG transformation for all values of κ. Therefore κ can be taken as a free parameter of the RG transformation. In the limit $\beta\to\infty$ the RG transformation which corresponds to the smearing out of δ-function (expression (46) for finite κ) can be written as

$$\begin{aligned}e^{-\beta' S'(\vec{\Phi}')}=\\ =\int_{\vec{\Phi}}\exp\left(-\beta\left\{S(\vec{\Phi})-\kappa\sum_{n_B}\left[\vec{\Phi}'_{n_B}\sum_{n\in n_B}\vec{\Phi}_n-|\sum_{n\in n_B}\vec{\Phi}_n|\right]\right\}\right)\end{aligned}\tag{47}$$

where $\beta'=\beta-\mathrm{O}(1)$, due to asymptotic freedom. Equation (47) is a saddle point problem in this limit, giving

$$S'(\vec{\Phi}')=\min_{\vec{\Phi}}\left\{S(\vec{\Phi})-\kappa\sum_{n_B}\left[\vec{\Phi}'_{n_B}\sum_{n\in n_B}\vec{\Phi}_n-|\sum_{n\in n_B}\vec{\Phi}_n|\right]\right\}.\tag{48}$$

The FP of the transformation is determined by the equation

$$S^*(\vec{\Phi}')=\min_{\vec{\Phi}}\left\{S^*(\vec{\Phi})-\kappa\sum_{n_B}\left[\vec{\Phi}'_{n_B}\sum_{n\in n_B}\vec{\Phi}_n-|\sum_{n\in n_B}\vec{\Phi}_n|\right]\right\}.\tag{49}$$

Equation (49) can be solved by stochastical methods for a given configuration $\vec{\Phi}'$. The final thing one has to do is to find a parameterization for the fixed point action.[4]

[4] Actually for Yang-Mills theories in 4 dimensions the real difficulty is to find a proper parameterization of the action with a finite number of parameters, whereas solving the fixed point equation for a given field configuration is a numerically feasible problem.

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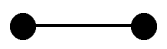
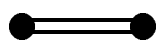
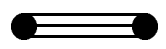
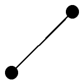
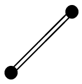
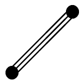
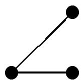
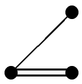
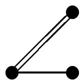
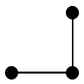
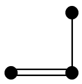
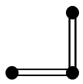
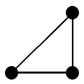
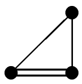
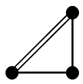
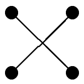
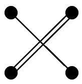
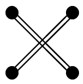
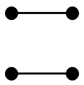
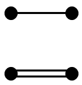
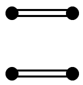
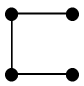
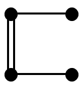
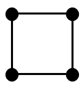
1		0.61884		−0.04957		−0.01163
4		0.19058		−0.02212		−0.00463
7		0.01881		−0.00139		0.00497
10		0.02155		0.00717		−0.00055
13		0.01078		0.00765		−0.00557
16		0.01209		−0.00114		0.00548
19		−0.00258		0.00387		−0.00100
22		−0.01817		−0.00772		0.04970

Table 2: The couplings used for the FP–action. The diagrams indicate the form of the interaction. All couplings used for this fit can be put on a 1x1 square. The coefficients of the quadratic and quartic interactions are calculated analytically, the higher order interactions are fitted.

We choose the parameterization

$$
\begin{aligned}
S^*(\vec{\Phi}) &= -\frac{1}{2}\sum_{n,r}\rho(r)\left(1-\vec{\Phi}_{n+r}\right) \\
&\quad + \sum_{n_1,n_2,n_3,n_4} c(n_1,n_2,n_3,n_4)\left(1-\vec{\Phi}_{n_1}\vec{\Phi}_{n_2}\right)\left(1-\vec{\Phi}_{n_3}\vec{\Phi}_{n_4}\right)+\ldots
\end{aligned}
\tag{50}
$$

where the summations go over all the lattice points. It is a significant help in the parameterization and optimalization problem that the first two functions ρ and c in equation (50) can be calculated analytically. $\rho(r)$ are just the coefficients for the perfect Laplace operator on the lattice as discussed in section 6. For the optimal choice $\kappa \approx 2$ $\rho(r)$ falls off as $\sim \exp(-3.44|r|)$. This is reasonable short ranged. The coefficients c have to be determined numerically. A good fit for the action can be obtained by using only 24 parameters. The values for an explicit parameterization of the fixed point action (50) are given in table 2.

7.5 Results obtained by simulating the FP action

The FP–action S^*_{FP} is defined as $\beta_{FP}S^*(\vec{\Phi})$. For large β_{FP}, the RT runs together with the FP–action (Fig.14), therefore S^*_{FP} is the perfect action. For intermediate β_{FP} values it is not perfect anymore, but – as we shall see – it performs amazingly well. As we discussed in the previous sections, the FP–action can be determined with the help of classical calculations. For $\kappa = 2$ it is very short ranged and a relatively simple parameterization describes it well. This parameterized form (given in Table 2) can be easily simulated. Our cluster Monte Carlo program performed very effectively. We would like to urge workers in this field to replace the standard action by the FP–action as a first approximation towards 'perfectness'.

We consider again the problem of the running coupling in the non-linear σ-model. We simulated the FP–action on a lattice of spatial size $L/a = 5$ and tuned β_{FP} until $m(L)L$ became close to the prescribed value: at $\beta_{FP} = 1.0821$ we got $g_r(L) = 1.0578(5)$. Then measuring on a lattice with $L'/a = 2L/a = 10$ at the same β_{FP} value we had $m(2L)2L = 1.2611(9)$. In the time direction the lattice was chosen to be at least 6–times larger than the finite box correlation length $\xi(L) = 1/m(L)$ and distances larger than three times ξ were used in the fitting procedure to obtain the mass gap. At the end we shifted $g_r(2L)$ according to the slight difference between the actual (1.0578) and prescribed (1.0595) value for $g_r(L)$ obtaining $g_r(2L) = 1.2638(12)$. For this shift we used the universal curve obtained in [Lüs91]. The error from this procedure is negligible. We repeated this calculation at $L = 10$ also, for $\beta_{FP} = 1.214$ with the results: $m(L)L = 1.0613(8)$ and $m(2L)2L = 1.2664(18)$. After shifting this gives $g_r(2L) = 1.2635(22)$. In Fig. 15 these two points are compared with the extrapolated prediction $g_r(2L) = 1.2641(20)$. No cut–off effects can be seen.[5]

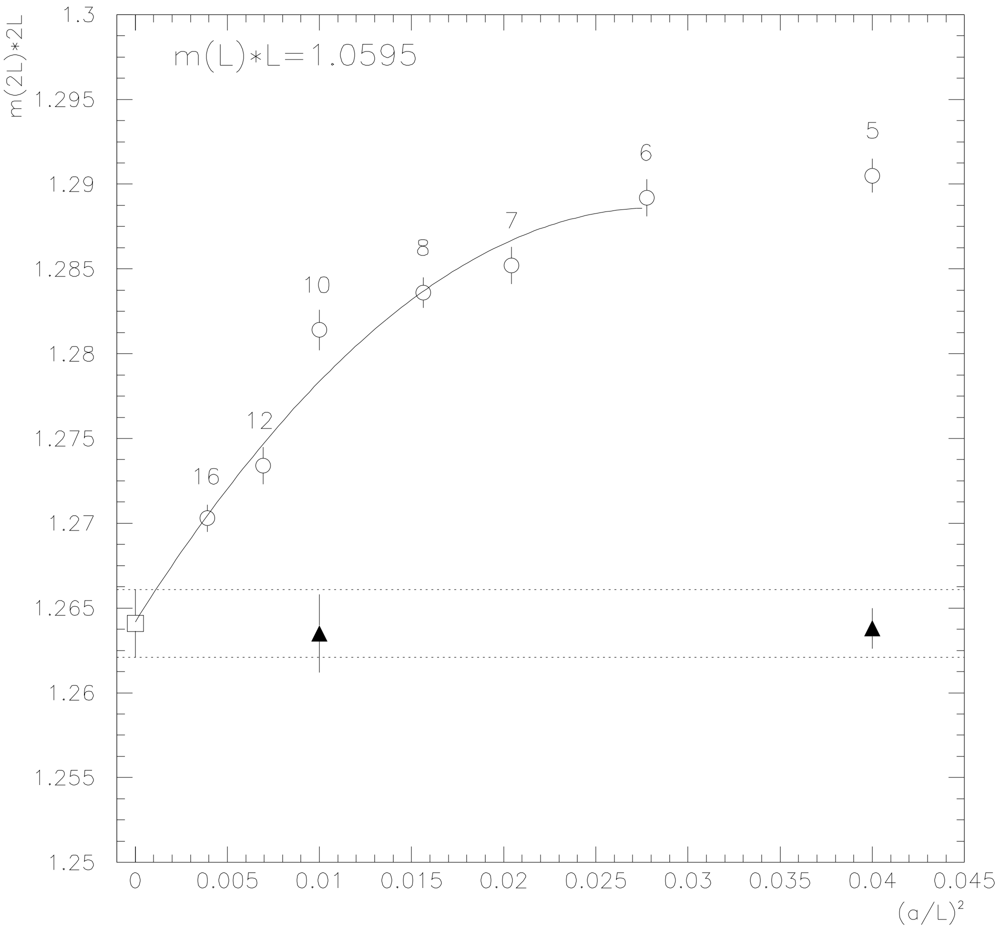


Figure 15: Simulating the perfect classical action.

[5]Surprisingly, even for an extremely coarse lattice, $L/a = 3$, (at $\beta_{FP} = 0.98$) one obtains $g_r(2L) = 1.2626(11)$, still with no sign of lattice artifacts.

List of Figures

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